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DIFFERENTIAL
EQUATIONS

H. BATEMAN



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DIFFERENTIAL EQUATIONS

BY

H. BATEMAN, PH.D., M.A.

LECTURER IN APPLIED MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE
FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE, AND READER IN
MATHEMATICAL PHYSICS IN THE UNIVERSITY OF MANCHESTER

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PREFACE

THE subject of Differential Equations has grown so rapidly in recent years that it is difficult to do justice to all branches of the subject in a single volume. In writing this book I have endeavoured to supply some elementary material suitable for the needs of students who are studying the subject for the first time, and also some more advanced work which may be useful to men who are interested more in physical mathematics than in the developments of differential geometry and the theory of functions. The chapters on partial differential equations have consequently been devoted almost entirely to the discussion of linear equations.

The order in which the material has been arranged is slightly different from that which is usually adopted. Instead of beginning with the standard forms of equations which can be solved very easily, I have devoted the second chapter to the method of integrating factors and the third to the method of transformation. A reader who is not already familiar with the subject will find it worth while to postpone the reading of §§ 13-17 and §§ 32-35 until he has mastered the more elementary work.

Chapters VII and VIII contain a few results which I think are new. In Chapter VI the treatment of special solutions bears some resemblance to a treatment which has just recently been sketched by Prof. Hill in an abstract of a memoir which will soon be published in the *Proceedings of the London Mathematical Society*. My work is independent of Prof. Hill's, for the manuscript was sent to the publishers at the beginning of 1915. Example 3, p. 143, was, however, added last September when marking the proof sheets, but these had been returned to the printers before I received my copy of Prof. Hill's abstract. Prof. Hill has evidently discussed the matter much more thoroughly than I have done, and I hope that some of my readers will consult his memoir.

I have naturally to acknowledge my indebtedness to previous books on differential equations, particularly to the works of Forsyth, Boole, Cohen, Wolsey Johnson and others too numerous to mention. I have also found Whittaker's *Analysis*, Carslaw's *Fourier Series and Integrals* and Bromwich's *Infinite Series* of great assistance in the preparation of Chapter IX. Some articles have been partly borrowed from Darboux' *Théorie of Surfaces*, and the treatises on analysis by the great French writers Jordan, Picard, Goursat and others have been invaluable to me.

I also wish to express my gratitude to my teachers at Cambridge, among whom I may mention Dr. Glaisher, who first roused my interest in the subject, Dr. Forsyth and Prof. Whittaker, who developed it by their lectures and writings. Some of their memoirs have been a source of inspiration to me in my research work.

I have great pleasure in thanking the officers and staff of the Glasgow University Press for the admirable work they have done in connection with the printing, and I am very grateful to the Editors of this Series of Mathematical Books for many useful suggestions.

HARRY BATEMAN.

November, 1916.

CONTENTS

CHAPTER I

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

SECTION	PAGE
1. Introduction - - - - -	1
2. The nature of the solution of an ordinary differential equation - - - - -	2
3. Supplementary conditions - - - - -	4
4. Equations in which the variables are separated - - -	10
5. Continuous and discontinuous solutions - - - -	13
6. A type of Green's function - - - - -	15

CHAPTER II

INTEGRATING FACTORS

7. Linear equations of the first order - - - - -	17
8. Equations reducible to linear equations - - - -	21
9. Integrating factors of two linear equations - - -	22
10. Systems of linear equations with constant coefficients -	24
11. Linear equations of the second order with constant coefficients - - - - -	28
12. Symbolical methods - - - - -	33
13. Discontinuous solutions of linear equations with constant coefficients - - - - -	38
14. A system of linear equations occurring in the theory of radio-active transformations - - - - -	41
15. A system of linear equations occurring in the theory of probability - - - - -	45
16. General theory of a system of linear equations - - -	46
17. Equations of the first order and of the first degree - -	48
18. Integrating factors of linear equations of the second order	53
19. Equations of various types - - - - -	56

CHAPTER III

TRANSFORMATIONS

SECTION		PAGE
20.	Preliminary remarks - - - - -	60
21.	Homogeneous equations of the first type - - - -	60
22.	Equations which can be reduced to homogeneous equations	63
23.	Clairaut's equation - - - - -	63
24.	Equations of the first order which can be solved for either x or y - - - - -	64
25.	Equations of the second order which do not contain x explicitly - - - - -	65
26.	Equations of the second order which do not involve y explicitly - - - - -	66
27.	Equations of the second order which may be solved by differentiation - - - - -	66
28.	Equations of Lagrange's type - - - - -	67
29.	Riccati's equation - - - - -	68
30.	The linear differential equation of the second order - -	71
31.	Equations reducible to linear equations with constant coefficients - - - - -	73
32.	Reduction of an equation of Laplace's type to a canonical form - - - - -	75
33.	Homogeneous equations of the second type - - - -	80
34.	The principle of duality and the theory of contact trans- formations - - - - -	81
35.	Jacobi's equation - - - - -	83

CHAPTER IV

GEOMETRICAL APPLICATIONS

36.	The differential equation of a family of algebraic curves	87
37.	Singular solutions of differential equations - - - -	88
38.	The c -discriminant - - - - -	92
39.	The p -discriminant - - - - -	94
40.	Differential equations which can be resolved into factors -	97
41.	Orthogonal trajectories - - - - -	98
42.	Orthogonal trajectories of a family of circles - - - -	100

CHAPTER V

DIFFERENTIAL EQUATIONS WITH PARTICULAR
SOLUTIONS OF A SPECIFIED TYPE

43.	Euler's equation - - - - -	103
44.	Linear differential equations with simple solutions - -	110

CONTENTS

ix

CHAPTER VI

PARTIAL DIFFERENTIAL EQUATIONS

SECTION	PAGE
45. Functional determinants and their properties - - -	116
46. Solution of the equation $J = 1$ - - -	119
47. The derivation of a partial differential equation from a given primitive - - -	121
48. Geometrical representation of the different types of primitives - - -	125
49. The characteristics of a partial differential equation of the first order - - -	131
50. Lagrange's linear equation - - -	134
51. Special solutions of Lagrange's equation - - -	140
52. The homogeneous linear equation - - -	143
53. Simple types of soluble partial differential equations. Standard forms - - -	144
54. The principle of duality - - -	148
55. Equations with three independent variables - - -	150

CHAPTER VII

TOTAL DIFFERENTIAL EQUATIONS

56. The condition of integrability - - -	153
57. Transformation of variables - - -	154
58. Invariance of the condition of integrability - - -	161
59. First method of solving the equation $\Lambda = 0$ - - -	161
60. Canonical form for Λ , second method of solution - -	164
61. Homogeneous equations - - -	167

CHAPTER VIII

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

62. The homogeneous linear partial differential equation of the second order - - -	169
63. Laplace's method - - -	173
64. D'Alembert's equation - - -	175
65. The equation of wave-propagation in three dimensions. Maxwell's equations - - -	179

SECTION	PAGE
65a. The fundamental equations of the theory of electrons -	186
66. Laplace's equation - - - - -	193
67. <u>The equation of the conduction of heat</u> - - - - -	211
68. Harmonic equations - - - - -	216
69. The condition that a given equation can be reduced to the harmonic form - - - - -	217

CHAPTER IX

INTEGRATION IN SERIES

70. Cauchy's existence theorem for an equation of the first order - - - - -	223
71. Runge's method of approximate solution - - - - -	227
72. The representation of solutions of linear differential equa- tions by means of power series - - - - -	230
73. The hypergeometric equation - - - - -	232
74. A limiting case of the hypergeometric equation - - - - -	236
75. Bessel's equation - - - - -	237
76. Legendre's equation - - - - -	240
77. Two convergence theorems - - - - -	242
78. The method of successive approximations - - - - -	245
79. The integration of linear partial differential equations by means of infinite series - - - - -	253

CHAPTER X

THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS
BY MEANS OF DEFINITE INTEGRALS

80. Laplace's transformation - - - - -	260
81. Equations of Pfaff's type - - - - -	263
82. Euler's transformation - - - - -	265
83. The solution of partial differential equations by means of definite integrals - - - - -	268
84. The Green's function of a linear differential equation with assigned boundary conditions - - - - -	275
85. Riemann's method - - - - -	280

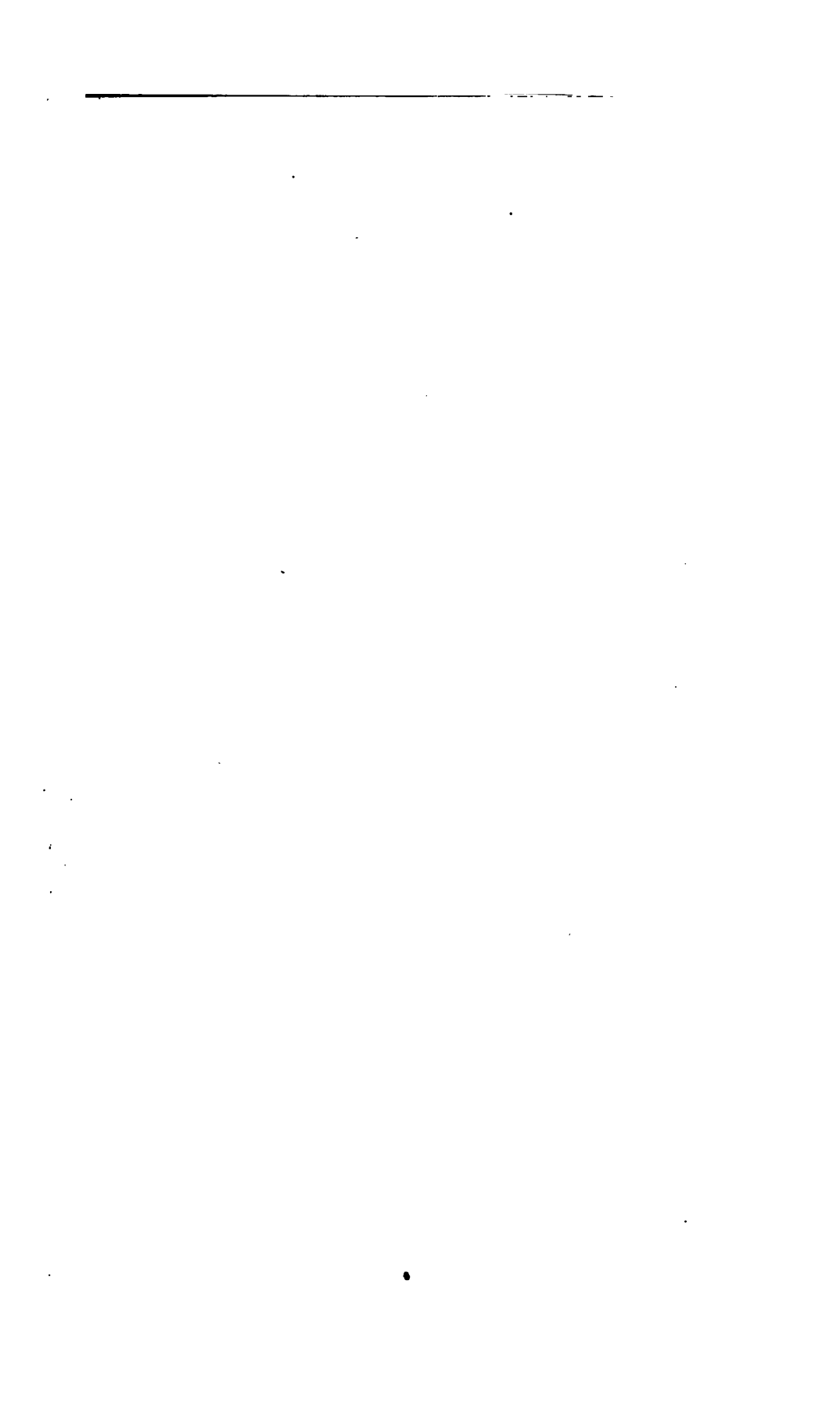
CONTENTS

xi

CHAPTER XI

THE MECHANICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

SECTION	PAGE
86. The reduction of a differential equation to a convenient form - - - - -	287
87. Mechanisms for the solution of equations of Pascal's type -	288
88. Mechanisms for integrating the product of two functions -	290
89. Pascal's mechanism for the solution of a linear equation of the first order - - - - -	293
90. The solution of a linear differential equation of the second order - - - - -	294
MISCELLANEOUS EXAMPLES - - - - -	296
INDEX - - - - -	304



CHAPTER I

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

§ 1. *Introduction.* When a mathematician attempts to describe natural phenomena with the aid of symbols, he frequently finds that he is unable to express the relation between two variable quantities x, t directly in the form $F(x, t) = 0$. The invention of the differential calculus has, however, led to a method by which the difficulty may sometimes be overcome. It is often easy to write down a relation between x, t and some of the derivatives of x with regard to t . Such a relation is a particular type of *differential equation*, and must be regarded as an auxiliary equation which expresses some but not all of the properties of the desired relation.

In general, a differential equation must be understood to mean a relation involving a derivative or derivatives of a function; it may involve the derivatives of other functions, and it may involve partial derivatives if there are several independent variables. If, however, the only derivatives which occur are partial derivatives, it is customary to call the equation a *partial differential equation*.

A differential equation which does not involve partial derivatives is frequently called an *ordinary differential equation*. Thus

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is an ordinary differential equation, and

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a partial differential equation. In the first case t is regarded as the *independent variable* and x as the *dependent variable*; in the second case, z is the dependent variable, x and y the independent variables. By making a slight change in the form in which a differential equation is presented, it is possible to make any one of the variables appear as the dependent variable; it is generally convenient to choose the dependent variable in such a way that the equation takes the simplest possible form. Thus the equation

$$\frac{d^2y}{dx^2} = f(x)$$

may also be written in the form

$$\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^3 f(x) = 0$$

but the first form is simpler and easier to deal with.

§ 2. *The nature of the solution of an ordinary differential equation.*

A relation between the dependent and independent variables is said to be a solution or integral of a differential equation if, when the dependent variable and its various derivatives are calculated from the relation and substituted in the differential equation, the latter becomes an identity; in other words, it is satisfied for all values of the independent variable. Thus $x = \sin t$ is a solution of the differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

because when we calculate $\frac{d^2x}{dt^2}$ from the given relation we find that

$$\frac{d^2x}{dt^2} = -\sin t = -x$$

and with this value of $\frac{d^2x}{dt^2}$ the differential equation is satisfied identically.

A differential equation of type

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}\right) = 0 \quad (3)$$

in which the highest derivative which occurs is of order n , is called a *differential equation of the n^{th} order*. If the equation can be thrown into such a form that F is a polynomial of the m^{th} degree in the quantity $\frac{d^nx}{dt^n}$, it is called a *differential equation of the m^{th} degree*.

Thus

$$\left(\frac{d^2x}{dt^2}\right)^3 \frac{dx}{dt} + \frac{d^2x}{dt^2} = 0$$

is a differential equation of the second order and of the third degree.

A differential equation generally possesses an infinite number of solutions, as might be inferred from an examination of a simple equation,

$$\frac{dx}{dt} = f(t)$$

which is satisfied by a relation of type

$$x = \int f(t) dt + c$$

where c is an arbitrary constant. An equation will be regarded as solved when an expression is obtained for the most general function which satisfies it, even if this expression involves integrals which

cannot be evaluated by the ordinary methods of the integral calculus.

In solving a differential equation of type (3), the chief aim is to obtain a class of relations of type

$$f(x, t) = 0$$

which is sufficiently general to include all possible solutions of the equation. It has been found by experience that if a solution involving n arbitrary constants can be found, it is generally possible to derive any particular solution from it either by making a special choice of the arbitrary constant or by some other artifice. Such a solution is called a *general solution* or *primitive* of the differential equation.

A solution involving n arbitrary constants may fail to be general if it can be expressed in terms of a fewer number of arbitrary constants; for instance, the primitive

$$x^2 + y^2 = a^2 + b^2$$

appears to involve two arbitrary constants a and b , but if we write $a^2 + b^2 = c^2$, an expression is obtained in which only one arbitrary constant, c , appears.

When a primitive involving n arbitrary constants is given, a differential equation of the n^{th} order can generally be derived from it by differentiating the given relation n times with regard to the independent variable x and then eliminating the n arbitrary constants from the n equations so obtained, making use of the original equation. No general method of performing this elimination can be given at this stage, but in a large number of cases a simple method of performing the elimination readily suggests itself.

Example 1. Find the differential equation corresponding to the primitive

$$y = \frac{x + a}{x + b}$$

where a and b are arbitrary constants.

First solve for a , and differentiate so as to eliminate a ; we get

$$\begin{aligned} yx + by - x &= a \\ \therefore x \frac{dy}{dx} + y + b \frac{dy}{dx} - 1 &= 0 \end{aligned}$$

Solving for b and writing $\frac{dy}{dx} = p$, we get

$$\begin{aligned} \frac{1}{p} - \frac{y}{p} - x &= b \\ \therefore -\frac{1}{p^2} \frac{dp}{dx} - 1 + \frac{y}{p^2} \frac{dp}{dx} - 1 &= 0 \end{aligned}$$

or

$$(y - 1) \frac{dp}{dx} = 2p^2$$

The differential equation is thus

$$(y - 1) \frac{d^2y}{dx^2} = 2 \left(\frac{dy}{dx} \right)^2$$

Example 2. Find the differential equation corresponding to the primitive

$$c^2x - c(x^2 + xy + y) + y(x + y) = 0$$

In this case the simplest plan is to differentiate the equation as it stands. Writing $\frac{dy}{dx} = p$, we get

$$c^2 - c(2x + xp + y + p) + (xp + y + 2yp) = 0$$

Eliminating c^2 , we find that

$$c(x^2 + x^2p + px - y) = x^2p + 2xyp - y^2$$

Also $c^2(x^2p + 2xyp - y^2)$

$$= c[p(x^3 + 2x^2y + xy^2 + y^2) + y^2 - 2xy^2 - x^2y - y^3]$$

$$\therefore (x^2p + 2xyp - y^2)^2 = [px(x + 1) + x^2 - y]$$

$$\times [p(x^3 + 2x^2y + xy^2 + y^2) + y^2 - 2xy^2 - x^2y - y^3]$$

$$\text{or } (p + 1)(px - y)(2x^2y + 2xyp - x^4 - 2x^3y - x^2y^2 - y^2) = 0$$

The differential equation is thus

$$\left(\frac{dy}{dx} + 1 \right) \left(x \frac{dy}{dx} - y \right) = 0$$

A solution involving the proper number of arbitrary constants may fail to give all possible solutions if the differential equation can be resolved into factors. For instance, the equation

$$\left(\frac{dy}{dx} - 1 \right) \left(\frac{dy}{dx} - 2 \right) = 0$$

is satisfied by $y = x + c$, and is of the first order. Consequently this solution involves the proper number of arbitrary constants, but there are solutions of type $y = 2x + a$ which are not included in the general relation $y = x + c$ and cannot be derived from it.

Some simple examples on the formation of differential equations are given on p. 16.

§ 3. *Supplementary conditions.* We have already seen that a differential equation does not specify completely the relation between two variables x and t . To complete the specification it is necessary to add certain supplementary conditions which are generally indicated by the circumstances in the problem to be solved. Sometimes the value of x is known for a particular value of t , and this suffices to determine the unknown constant. Sometimes the fact that x is real when t is real becomes of importance, while the condition that x is finite for all real positive values of t frequently

occurs and leads to the determination of a constant. It may happen that there is more than one way of choosing the arbitrary constants in the solution so as to make x a *real* function of t , and in this case the *real* relations to which the differential equation is equivalent form a number of classes instead of only one. Again, it may happen that there is only one real solution of a differential equation or none at all.

The following examples will illustrate the different points :

1°. A general solution of the equation

$$\frac{dy}{dt} = \cos t$$

is given by

$$y = \sin t + c$$

If, however, a supplementary condition states that $y = 0$ when $t = 0$, the constant c is completely determined, and we have $y = \sin t$.

2°. The equation $\left(\frac{dx}{dt} - 1\right)^2 + (x - t)^2 = 0$

possesses only one real solution, namely $x = t$.

3°. The equation $\left(\frac{dx}{dt}\right)^2 + (x - t)^2 = 0$

does not possess a real solution.

4°. The equation $t \frac{dx}{dt} + x^2 = 0$

possesses the general solution $x = \frac{1}{c + \log t}$, but the condition $x = 0$ when $t = 0$ is not sufficient to determine the unknown constant c , for the condition is always satisfied.

5°. If a is a positive constant, the equation

$$\frac{dx}{dt} = a^2 - x^2$$

possesses two distinct types of real solutions.

A solution is given by

$$t = \int \frac{dx}{a^2 - x^2} + \text{const.},$$

and the integral on the right-hand side has different real forms according as x^2 is greater or less than a^2 .

We may write $t - t_0 = \frac{1}{2a} \log \frac{a+x}{a-x} \quad -a < x < a$

$$t - t_0 = \frac{1}{2a} \log \frac{x+a}{x-a} \quad x^2 > a^2$$

where t_0 is an arbitrary real constant.

In the first case,
$$x = a \frac{e^{2at} - e^{2at_0}}{e^{2at} + e^{2at_0}}$$

while in the second case,
$$x = a \frac{e^{2at} + e^{2at_0}}{e^{2at} - e^{2at_0}}$$

In each case there is a real value of x for any real value of t , but if the two equations are interpreted kinematically, the two representative motions are essentially distinct in character. If the supplementary condition is added that x must be finite for all real values of t , the solution is necessarily of the first type. If $x = c$ when $t = 0$, the motion is of the first type if $c^2 < a^2$, and of the second type if $c^2 > a^2$. If $x = a$ when $t = 0$, we must have $t_0 = -\infty$, and both equations give $x = a$.

If we use the symbol $|x - a|$ to denote the *modulus* or absolute value of $x - a$, the two real solutions may be embodied in the single formula

$$t - t_0 = \frac{1}{2a} \log \left| \frac{x + a}{x - a} \right|$$

In formulating a differential equation corresponding to a given problem, the dependent and independent variables should be chosen so that the differential equation is as simple as possible; care should also be taken to include all the conditions and restrictions peculiar to the problem in the supplementary conditions; note should be made, for instance, of the range of values of the independent variable (or some other quantity) for which the problem is a possible one. Some examples will now be given for the sake of illustration.

EXAMPLES.

1. A smooth board with a rough curved edge is placed on a smooth table with its curved edge resting against a straight ledge of length $2a$. A light string, carrying a weight W at each end, passes round a nail fixed in the board at C and over the ends of the ledge, the weights

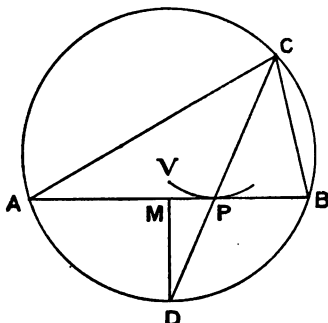


FIG. 1.

being suspended in mid air. Find the form of the curved edge so that the board may be in neutral equilibrium.

Let A, B be the ends of the ledge, P the point of contact and M the middle point of AB. For equilibrium, the resultant of the two equal tensions at C must pass through P and make with the normal at P to AB an angle which is not greater than the angle of friction. The resultant bisects the angle at C, and so we must have $\hat{ACP} = \hat{PCB}$. Let $MP = s$; then the equilibrium will be neutral if the board is still in equilibrium after it is rolled a small distance ds along AB. The condition for rolling is that s should be always equal to the length of the arc of the edge measured from some fixed point V which comes in contact with M.

Let us take s as the independent variable, and $r = CP$ as the dependent variable. If the angle CPB be denoted by ϕ , we have $\cos \phi = -\frac{dr}{ds}$; we must now endeavour to express $\cos \phi$ in terms of r and s .

Let the circle ACB meet CP in D; then, since the angle ACD is equal to the angle BCD, it follows that the arc AD equals the arc BD, and so DM is at right angles to AB.

$$\text{Now} \quad \cos \phi = \frac{PM}{PD}$$

$$\text{and} \quad PD \cdot PC = PA \cdot PB = (a + s)(a - s)$$

$$\text{therefore} \quad \cos \phi = \frac{rs}{a^2 - s^2}$$

The required differential equation is thus

$$-\frac{dr}{ds} = \frac{rs}{a^2 - s^2}$$

Writing this in the form

$$-\int \frac{dr}{r} = \int \frac{s ds}{a^2 - s^2}$$

$$\text{we see that} \quad \log r = \frac{1}{2} \log (a^2 - s^2) + \text{const.}$$

$$\text{or} \quad r = b \sqrt{a^2 - s^2}$$

To determine the constant, let us suppose that $CV = c$; then $r = c$ when $s = 0$. It should be noticed that when $s = 0$, $\phi = \frac{\pi}{2}$, and so CV is a normal to the curve, also $\frac{dr}{ds}$ is negative when s is positive; hence, if the curve be regarded as symmetrical* about CV, c is the greatest distance of C from the edge.

Putting $r = c$ when $s = 0$, we get $c = ab$. Therefore

$$r = \frac{a}{c} \sqrt{a^2 - s^2}$$

$$\text{or} \quad s^2 = a^2 \left(1 - \frac{r^2}{c^2} \right)$$

* In this case s must be regarded as positive whether it is measured to the left or right of M and V.

To find the polar equation of the curve referred to C as origin, we make use of the equation

$$\begin{aligned} r^2 \left(\frac{d\theta}{dr} \right)^2 &= \left(\frac{ds}{dr} \right)^2 - 1 \\ &= \frac{(s^2 - a^2)^2}{r^2 s^2} - 1 = \frac{(a^2 + c^2)r^2 - c^4}{c^2(c^2 - r^2)} \end{aligned}$$

$$\therefore \theta = - \int \frac{dr}{cr} \sqrt{\frac{(a^2 + c^2)r^2 - c^4}{c^2(c^2 - r^2)}} + \alpha$$

where α is a constant. Putting $t^2(c^2 - r^2) = r^2(a^2 + c^2) - c^4$, we get

$$\theta - \alpha = \int_t^\infty \left[\frac{a^2 + c^2}{a^2 + c^2 - t^2} - \frac{c^2}{c^2 + t^2} \right] \frac{dt}{c}$$

Now, if $\theta = 0$ when $r = c$, i.e. when $t = \infty$, we have $\alpha = 0$, and so the polar equation of the curve is

$$\theta = \sqrt{\left(1 + \frac{a^2}{c^2}\right)} \cot^{-1} \sqrt{\frac{a^2 c^2}{(a^2 + c^2)(c^2 - r^2)}} - 1 - \cot^{-1} \sqrt{\frac{a^2 r^2}{c^2(c^2 - r^2)}} - 1$$

It should be remarked that only a portion of the edge of the board can have this shape; moreover, there can only be equilibrium when $\frac{\pi}{2} - \varphi$ is not greater than the angle of friction. If ϵ is the angle of friction, we must have $\cos \varphi \leq \sin \epsilon$, that is

$$\frac{cs}{a\sqrt{a^2 - s^2}} \leq \sin \epsilon$$

This inequality is satisfied for values of s lying between 0 and σ , where

$$\frac{c\sigma}{a\sqrt{a^2 - \sigma^2}} = \sin \epsilon$$

It is easy to see that σ lies between 0 and a . If σ is given, the value of c can be determined from the above equation.

2. Three boys A, B, C running at the same speed u , start to chase one another. A pursues B, B pursues C and C pursues A.

Find a differential equation which will indicate the way in which the ratios of the sides of the triangle ABC vary.*

Let a, b, c be the lengths of the sides of the triangle ABC at time t ; A, B, C the angles opposite these sides; then

$$\frac{da}{dt} = -u(1 + \cos C) = -2u \frac{s(s-c)}{ab}$$

where

$$2s = a + b + c$$

$$\text{Similarly} \quad \frac{db}{dt} = -2u \frac{s(s-a)}{bc} \quad \frac{dc}{dt} = -2u \frac{s(s-b)}{ca}$$

* This problem is due to Prof. Morley; his differential equation is discussed in a paper by F. E. Hackett, *Johns Hopkins Circular*, July, 1908.

$$\begin{aligned}
 \therefore \frac{d(s-a)}{dt} &= -\frac{u}{abc} [s^2(a+b-c) - s(a^2 + b^2 - c^2)] \\
 &= -\frac{u}{abc} [2s^2(a+b-s) - s(a^2 + b^2) + s(2s-a-b)^2] \\
 &= -\frac{2us(s-a)(s-b)}{abc} = k(s-a)(s-b), \text{ say}
 \end{aligned}$$

Now write $s-a = \alpha$, $s-b = \beta$, $s-c = \gamma$; then

$$\frac{d\alpha}{dt} = k\alpha\beta \quad \frac{d\beta}{dt} = k\beta\gamma \quad \frac{d\gamma}{dt} = k\gamma\alpha$$

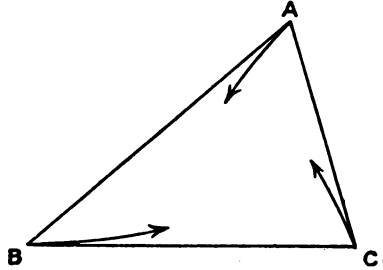


FIG. 2.

Putting $\beta = \gamma x$, $\alpha = \gamma y$, we get

$$\begin{aligned}
 \frac{d\beta}{dt} &= x \frac{d\gamma}{dt} + \gamma \frac{dx}{dt} & \frac{d\alpha}{dt} &= y \frac{d\gamma}{dt} + \gamma \frac{dy}{dt} \\
 \therefore \frac{dy}{dx} &= \frac{k\beta\gamma - kx\gamma\alpha}{k\alpha\beta - k\gamma\gamma\alpha} = \frac{x - xy}{xy - y^2}
 \end{aligned}$$

Hence we finally obtain the differential equation

$$\frac{dy}{dx} = \frac{x - xy}{cy - y^2}$$

As regards the supplementary conditions, we note in the first place that since $b+c < a$, $c+a < b$, $a+b < c$, α , β , γ are never negative, consequently x and y are never negative.

If we have initially $a = a_0$, $b = b_0$, $c = c_0$, $s = s_0$, the other supplementary condition is

$$y = \frac{s_0 - a_0}{s_0 - c_0} \text{ when } x = \frac{s_0 - b_0}{s_0 - c_0}$$

It is easy to see that if the boys are treated simply as points, there will be no proper capture; in fact, each boy will describe a kind of spiral curve. If, indeed, we suppose that A captures B and is not at the same time captured by C, we are soon led to a contradiction. We should have, in fact, $c = 0$, $a = b \neq 0$. Hence $\alpha = \beta = 0$, $\gamma \neq 0$. These equations give $x = 0$, $y = 0$, while, on the other hand,

$$\frac{y}{x} = \frac{\alpha}{\beta} = \frac{s-a}{s-b} = \frac{\tan \frac{1}{2}B}{\tan \frac{1}{2}A} = \frac{\tan \frac{1}{2}B}{\tan \left(\frac{\pi}{2} - \frac{B}{2} - \frac{C}{2} \right)}$$

Consequently, if A is running in a definite direction relative to BC when c is zero, the limiting value of $\frac{y}{x}$ is $\tan^2 \frac{1}{2}B$. This must also be the limiting value of $\frac{dy}{dx}$, but the expression $\frac{x - xy}{xy - y^2}$ for $\frac{dy}{dx}$ is infinite when $x = 0$ and $y = 0$; consequently we are led to a contradiction unless $\tan^2 \frac{1}{2}B$ is infinite, and in this case the three boys would be in one straight line just before B is captured. This, however, is also impossible (Fig. 3), because, since the angle at B is 180° , A and B would

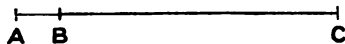


FIG. 3.

be running in the same direction, and since they are running at the same speed, A could not capture B; moreover, if the boys are in one straight line at any instant, they must always have been in a straight line, and this is inconsistent with the initial condition that they form a triangle.

3. A runs in a straight line with velocity u and B pursues A with velocity v , steering his course so that he is always running directly towards A. Find a differential equation for the path of B.

4. Find a differential equation for the path of B when A runs in a circle instead of in a straight line.

5. Find the differential equation giving the motion of a particle in a smooth straight tube, which revolves in a horizontal plane with constant angular velocity ω , one point of the tube being fixed.

6. A ray of light which starts from a fixed point S is always reflected at a curve C so that it passes through another fixed point S'. Find a differential equation to determine the class of curves to which C belongs, and write down the supplementary condition (1) when we are given one of the points in which the curve C meets SS', and, secondly, when we are given one of the points in which the curve C meets the line bisecting SS' at right angles. Show from geometrical considerations that in the second case a supplementary condition involving only real quantities implies that the required solution of the differential equation belongs to one of two classes into which the solutions of the differential equation can be divided.

§ 4. *Equations in which the variables are separated.*

A differential equation of type

$$\frac{dx}{dt} = \frac{f(x)}{g(t)}$$

may be solved at once by writing

$$\int \frac{dx}{f(x)} = \int \frac{dt}{g(t)} + c$$

In particular, the solution of the equation

$$\frac{dx}{dt} = f(x)$$

is

$$t - t_0 = \int \frac{dx}{f(x)}$$

where t_0 is an arbitrary constant.

It is generally convenient to have an expression for x as a function of t , and to obtain this we must reverse the preceding equation.

If, for instance, the rectilinear motion of a particle is specified by the equation

$$\frac{dx}{dt} = \sqrt{a^2 - x^2}$$

where a is a constant, we have

$$t - t_0 = \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Reverting the equation, we obtain

$$x = a \sin (t - t_0)$$

and the periodicity of the motion is manifest.

An important series of equations of the present type occurs in Chemical Dynamics, in the applications of the law of mass action.

This law states that if in a chemical reaction n substances are being transformed into new substances, the rate at which the change takes place is proportional to the product of the active masses of the original substances present at that instant.*

Thus, if $\lambda a, \mu b, \nu c, \dots$ denote the masses of the original substances at the beginning of the reaction and $\lambda x, \mu x, \nu x$ the amounts of these substances that have been transformed at time t , the law of mass action states that if the velocity of reaction in the reverse direction may be neglected,

$$\frac{d}{dt}(\lambda x) = K\lambda(a - x)\mu(b - x)\nu(c - x) \dots$$

where K is a constant. This equation may be written in the form

$$\frac{dx}{dt} = k(a - x)(b - x)(c - x) \dots$$

where k is a constant.

EXAMPLES †

1. The equation of a unimolecular chemical reaction is

$$\frac{dx}{dt} = k(a - x)$$

where a is the initial amount of the original substance, x is the amount

* For further information the reader is referred to J. W. Mellor's *Chemical Statics and Dynamics*.

† Most of these have been taken from Lamb's *Infinitesimal Calculus*.

transformed at the end of time t and k is a positive constant. Obtain an expression for x , using the supplementary condition that $x = 0$ when $t = 0$.

2. Discuss the equations

$$\frac{dx}{dt} = k(a - x)(b - x)$$

$$\frac{dx}{dt} = k(a - x)^2$$

regarding them as equations of bimolecular chemical reactions.

3. A particle falls through a resisting medium in which the resistance is proportional to the square of the velocity. Show that the equation of motion is

$$\frac{dv}{dt} = g - kv^2$$

where k is a constant and g is the acceleration due to gravity, and is also supposed to be constant. Integrate this equation, and show that as t increases without limit the velocity v tends to the value $\sqrt{\frac{g}{k}}$.

4. The distance x from the axis of a thick cylindrical tube of metal is related to the internal pressure p as indicated in the equation

$$\frac{dp}{dx} = \frac{a - 2p}{x}$$

where a is a constant. Hence show that $p = \frac{1}{2}a + Cx^{-2}$, where C is a constant to be determined by the supplementary condition.

5. The equation of rectilinear motion of a particle under an attractive force varying inversely as the square of the distance from a fixed point is

$$v \frac{dv}{dx} = -\frac{\mu}{x^2}$$

where v is the velocity of the particle when it is at a distance x from the fixed point and μ is a positive constant.

Prove that if the particle starts from rest at a distance $2a$ from the fixed point, the motion is represented by the equations

$$x = 2a \cos^2 \theta \qquad t = a^{\frac{3}{2}} \mu^{-\frac{1}{2}} (2\theta + \sin 2\theta)$$

6. If in the last example the particle has an initial velocity u when at a distance c from the fixed point, the equations defining the motion have different forms according as u^2 is greater, equal to, or less than $2\mu/c$.

7. The form of a heavy chain of uniform density hanging from two fixed points is determined by the condition that the centre of gravity of any portion is vertically above the point of intersection of the tangents at the two ends. If s denote the length of an arc of the chain measured from a point at which the tangent is horizontal, the form of the chain is determined by the differential equation

$$\frac{d}{dx}(\log s) = \frac{d}{dx} \left(\log \frac{dy}{dx} \right)$$

Hence show that the form of the chain is given by the equations

$$s = c \sinh \frac{x}{c} \qquad y = c \cosh \frac{x}{c} + a$$

where c and a are constants.

8. In a suspension bridge with uniform horizontal load the form of the chain is determined by the condition that any two tangents to the curve intersect on the vertical line bisecting the chord of contact. Prove that the curve formed by the chain must be a parabola with its axis vertical.

9. Find a curve in which the tangent and normal at each point form a triangle of constant area with a fixed line.

10. Find a curve in which the projection of the normal on the radius vector from the origin is of constant length.

(*Hint.* Obtain a differential equation in which the coordinate x and the angle made by the radius vector with the axis of x are variables. The normal is supposed to be terminated by the axis of x and the curve.)

11. A rotating fly-wheel is being stopped by a fluid resistance which decreases the angular momentum at a rate $F\omega$, where ω is the angular velocity of the fly-wheel at time t . If I is the moment of inertia of the fly-wheel about its axis of rotation, find how long the resistance must act to reduce the angular velocity from ω to $\frac{\omega}{2}$.

§ 5. Continuous and discontinuous solutions.*

Let us now consider the differential equation

$$\frac{d^2x}{dt^2} = 0$$

which when interpreted kinematically gives a partial description of the motion of a particle which moves under forces whose components parallel to the axis of x are constantly zero.

We shall find it convenient to consider two cases.

1°. When the particle does not collide with any obstacles, successive integrations give

$$\frac{dx}{dt} = v$$

$$x = x_0 + vt$$

where x_0 and v are arbitrary constants. If we are given the initial position and initial velocity of the particle, x_0 and v can be found in just one way, and the motion is completely determined. In this case x and $\frac{dx}{dt}$ are continuous functions of t for all values of t .

2°. When the particle suffers a number of collisions in the course of its motion. The derivative $\frac{dx}{dt}$ is in this case a discontinuous

* The term discontinuous solution is used here to mean a solution which is either a discontinuous function itself or whose derivatives are not all continuous functions.

function of t , and the value of v in the preceding equations is different for different values of t .

To obtain a convenient expression for the relation between x and t in this case, we make use of the fact that if t_1 is a constant, the function

$$x = |t - t_1|$$

is a solution of the differential equation. With this value of x the derivative changes from -1 to $+1$ as t passes through the value t_1 . To obtain a solution corresponding to the case in which the x -component of the particle's velocity changes suddenly from v to $-u$ as x passes through the value t_1 , we write

$$x = a + \frac{1}{2}(v - u)t - \frac{1}{2}(v + u)|t - t_1|$$

where a is an arbitrary constant which may be determined from the initial conditions. The case in which the derivative suffers a number of discontinuities may be dealt with by using a number of terms of type $|t - t_1|$.

Sometimes the supplementary conditions are incompatible with the continuity of x and its derivative. For instance, if we are given that $\frac{dx}{dt} = v$ for $t = 0$ and $\frac{dx}{dt} = -v$ for $t = 1$, it is not possible to obtain a relation of type $x = x_0 + ut$ which will hold for all values of t in the interval $0 \leq t \leq 1$.

Sometimes the supplementary conditions imply that if x and $\frac{dx}{dt}$ are continuous, they are identically zero. For instance, if we are given that $x = 0$ when $t = 0$ and when $t = 1$, the equation $x = x_0 + vt$ implies that $x_0 = 0$, $v_0 = 0$, and consequently $x \equiv 0$.

The supplementary conditions can, however, be satisfied by a non-vanishing function x , whose derivative is discontinuous. To make the relation between x and t quite definite, we shall add the further condition that $\frac{dx}{dt}$ decreases in magnitude by unity as t passes through the value t_1 .

The relation must evidently be of the form

$$x = a + bt - \frac{1}{2}|t - t_1|$$

where a and b are constants at our disposal. The supplementary conditions give

$$0 = a - \frac{1}{2}t_1, \quad 0 = a + b - \frac{1}{2}(1 - t_1)$$

$$\text{Hence} \quad x = \frac{1}{2}(t + t_1) - tt_1 - \frac{1}{2}|t - t_1|$$

It should be noticed that x is a *symmetric function* of t and t_1 ; in other words, the value of x is unaltered when t and t_1 are interchanged.

Let us denote this function by $g(t, t_1)$; then it is easily seen that

$$\begin{aligned} x = g(t, t_1) &= t(1 - t_1) \quad \text{when } 0 \leq t \leq t_1 \\ &= t_1(1 - t) \quad \text{,, } t_1 \leq t \leq 1 \end{aligned}$$

§ 6. *A type of Green's function.* The function $g(t, t_1)$ occurs naturally when we try to find a solution of

$$\frac{d^2x}{dt^2} + f(t) = 0 \quad (1)$$

and the supplementary equations $x = 0$ when $t = 0$ and $t = 1$, on the understanding that $\frac{dx}{dt}$ is to be continuous throughout the interval $(0 \leq t \leq 1)$. Let us assume that $f(t)$ is continuous throughout the same interval; then we find by direct integration that

$$\frac{dx}{dt} + \int_0^t f(t_1) dt_1 = A$$

where A is a constant. A second equation may be obtained by using the fact that $t \frac{d^n x}{dt^n}$ can be integrated directly if r is an integer less than n . Multiplying equation (1) by t and integrating, we obtain

$$t \frac{dx}{dt} - x + \int_0^t f(t_1) t_1 dt_1 = B$$

where B is a constant. Hence

$$x = \int_0^t (t_1 - t) f(t_1) dt_1 + At - B$$

The supplementary conditions give $B = 0$ and

$$A = \int_0^1 (1 - t_1) f(t_1) dt_1$$

consequently

$$x = \int_0^t (t_1 - t) f(t_1) dt_1 + \int_0^1 t(1 - t_1) f(t_1) dt_1$$

or

$$x = \int_0^1 g(t, t_1) f(t_1) dt_1$$

The required solution of the differential equation is thus expressed as a definite integral taken over the range $(0, 1)$. The function $g(t, t_1)$ occurring in this integral is called the *Green's function* of the differential expression $\frac{d^2x}{dt^2}$ and the given supplementary conditions; it was so named by H. Burkhardt in 1894, because it is to some extent analogous to the function used by G. Green in his famous memoir on electricity and magnetism (1828).

EXAMPLES.

1. Prove that a solution of the equation

$$\frac{d^n x}{dt^n} = f(t)$$

is given by the formula

$$x = \int_s^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau) d\tau + A_0 + A_1(t-s) + \dots + A_{n-1} \frac{(t-s)^{n-1}}{(n-1)!}$$

where s is a constant and $A_0, A_1, \dots, A_{n-1}$ are the values of the derivatives $x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}$ when $t=s$.

2. Prove that if $x=0$ when $t=0$, $\frac{dx}{dt}=0$ when $t=1$, and

$$\frac{d^2x}{dt^2} + f(t) = 0$$

The only function which satisfies these conditions and has a continuous derivative for every point of the interval $(0, 1)$ is given by the formula

$$x = \int_0^1 G(t, t_1) f(t_1) dt_1$$

where

$$\begin{aligned} G(t, t_1) &= t \quad \text{when } t \leq t_1 \\ &= t_1 \quad \text{when } t \geq t_1 \end{aligned}$$

3. A ray of light passes through a medium in which v , the velocity of light, depends only on the distance y from a fixed plane. If ψ is the angle which the tangent to the ray makes with the fixed plane, the law of refraction is $v \sec \psi = \text{const.} = k$, say. Prove that if v increases with y and surpasses the value k , the differential equation for the path of the ray should be written in the form

$$v \left| \frac{dy}{dx} \right| = +\sqrt{k^2 - v^2}$$

for $\frac{dy}{dx}$ has different signs on the two sides of the vertex of the ray.

4. Find the differential equations * corresponding to the following primitives, a, b and c being arbitrary constants:

$$\begin{array}{ll} (x-c)(y-c) = 1 & (x-a)^2 + (y-b)^2 = 1 \\ y^2 = 4ax + b & y = (c-x)^3 \\ y = a \sin(bx+c) & x^3 + y^3 + 3axy = a^3 \\ y = cx - c^3 & (y-b)^2 = 4ax \end{array}$$

5. The primitive $y = ae^{bx+c}$ leads to a differential equation of the second order, while the primitive

$$y = a^{bx}$$

leads to an equation of the first order. Explain why the order is less than the apparent number of arbitrary constants in these cases.

* Of course innumerable differential equations can be obtained from a given primitive by eliminating the constants, but in this example the equation of lowest order is required in each case.

CHAPTER II

INTEGRATING FACTORS

§ 7. *Linear Equations of the First Order.* Let us consider the differential equation

$$\frac{dy}{dx} + Py = Q \dots \dots \dots (1)$$

in which P and Q are any integrable functions of x . This equation is said to be *linear*, because each term is at most of the first degree in the function y and its derivative ; it is said to be of the *first order*, because the first derivative of y is the highest derivative occurring in the equation.

To solve the equation we shall endeavour to find an *integrating factor* v . This is defined to be a function of x such that

$$v \left(\frac{dy}{dx} + Py \right) \equiv \frac{d}{dx} (vy) \dots \dots \dots (2)$$

If a suitable function $v(x)$ can be found, the differential equation may be written in the form

$$\frac{d}{dx} (vy) = Qv \dots \dots \dots (3)$$

any may be integrated at once, giving

$$vy = \int Qv dx + C \dots \dots \dots (4)$$

where C is an arbitrary constant.

To find a suitable value of v , we equate the coefficients of y in (2), and obtain

$$\frac{dv}{dx} = Pv$$

Writing this equation in the form

$$\frac{1}{v} \frac{dv}{dx} = P$$

and integrating, we obtain

$$\log v = \int P dx + A$$

where A is a constant. Since a particular integrating factor will serve our purpose, we may put $A = 0$, and write

$$v = \exp \left(\int P dx \right)$$

The general value of v is obtained by multiplying this value by an arbitrary constant ; if we use this general value in (4), we simply obtain the same result as if we used the particular value.

Substituting the particular value of v in (4), we obtain

$$y = C \exp \left(- \int P dx \right) + \exp \left(- \int P dx \right) \int Q \exp \left(\int P dx \right) dx$$

This is the *general solution* of the equation. The constant C can usually be determined with the aid of a supplementary condition. If, for instance, we are given that $y = b$ when $x = a$ and $v(x)$ is not infinite or indeterminate when $x = a$, the value of C may be determined uniquely by putting $x = a$ in equation (4). The complete solution may be written in the form

$$v(x)y - v(a)b = \int_a^x Q(t)v(t) dt$$

If a *maximum* or *minimum* value of y is known to be a given real quantity h , there may be several possible values for the constant C .

In fact, if we put $y = h$, $\frac{dy}{dx} = 0$ in equation (1), we obtain an equation

$$P(x)h = Q(x)$$

which may determine several real values of x or none at all. If $x = a$ is one of the values of x for which the equation is satisfied, an appropriate constant C may be obtained by putting $x = a$, $y = h$ in equation (4).

If we are given the *mean value* of y for a certain range of values of x , the constant C can be determined uniquely, provided the mean value of $\frac{1}{v}$ is not zero. In the alternative case the supplementary condition is incompatible with the differential equation, unless the assigned mean value M is also the mean value of the quantity

$$\frac{1}{v} \int Qv dx$$

When this condition is satisfied, the constant C may be chosen arbitrarily, and so every solution of the differential equation satisfies the supplementary condition. Thus, when the mean value of $\frac{1}{v}$ is zero, the problem of finding an appropriate solution is *poristic* ; in other words, there is either no solution at all or an infinite number of solutions.

If we are given the *mean value of y^2* over a certain range, it follows from equation (4) that there are generally two values of C which may or may not be real. Hence, in this case, the differential equation possesses an appropriate real solution only when a certain inequality is satisfied.

It should be noticed that the *difference equation* $y_{n+1} + p_n y_n = q_n$, in which n is a positive integer and p_n, q_n given functions of n , may be solved in a similar way. An integrating factor u_n must now be chosen so that the equation may be written in the form $u_n y_{n+1} - u_{n-1} y_n = u_n q_n$. This requires that $u_{n-1} = -u_n p_n$. Solving this equation, we get $u_n = (-1)^n \frac{u_0}{p_1 p_2 \dots p_n}$, where u_0 is an arbitrary constant which may be taken to be unity. The solution of the original equation in its new form is obtained at once by addition of the equations in which n has successive integral values, and we obtain $u_n y_{n+1} - u_0 y_1 = u_1 q_1 + u_2 q_2 + \dots u_n q_n$ where y_1 is an arbitrary constant.

EXAMPLES.

1. If i is the strength of the electric current flowing along a uniform wire at time t , L the coefficient of self-induction of the wire, R the resistance and E the electromotive force generating the current, we have

$$L \frac{di}{dt} + Ri = E$$

(The first term is analogous to mass $\times$ acceleration and the second to a frictional force proportional to the velocity.)

We shall integrate this equation on the supposition that $E = E_0 \sin \omega t$ and that E_0, ω, R, L are constants.

The equation determining an integrating factor v is $L \frac{dv}{dt} = Rv$, hence we may take

$$v = e^{\frac{Rt}{L}}$$

Writing the equation in the form

$$\frac{d}{dt} \left(L i e^{\frac{Rt}{L}} \right) = E_0 e^{\frac{Rt}{L}} \sin \omega t$$

and integrating, we get

$$L i e^{\frac{Rt}{L}} = L E_0 \left(\frac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2 L^2} \right) e^{\frac{Rt}{L}} + C$$

where C is an arbitrary constant. To simplify this result, we write $\omega L = S \sin \varphi$, $R = S \cos \varphi$, $AL = CS$, where A is a new arbitrary constant. Hence, finally,

$$Si = E_0 \sin (\omega t - \varphi) + A e^{-\frac{Rt}{L}}$$

where

$$S^2 = R^2 + \omega^2 L^2 \quad \tan \varphi = \frac{\omega L}{R}$$

If the supplementary condition states that $i = i_0$ when $t = 0$, we have

$$A = Si_0 + E_0 \sin \phi$$

It should be noticed that if this initial value i_0 is such that

$$Si_0 + E_0 \sin \phi = 0$$

we have $A = 0$, and the equation possesses a periodic solution i of period $T = \frac{2\pi}{\omega}$, i.e. the period of the generating electromotive force.

Physical considerations indicate that L and R are both positive, hence the term $Ae^{-\frac{Rt}{L}}$ is very small when t is large and positive; consequently the value of i ultimately differs very little from the periodic value whatever may be the initial condition.

2. If in the last example the mean value of i over an interval of length T is zero, $A = 0$, and the solution is periodic.

3. If the mean value of i^2 over the same interval is equal to a^2 , we have

$$S^2 a^2 = \frac{1}{2} E_0^2 + \frac{A^2 L}{2R} e^{-\frac{2R\tau}{L}} \left(1 - e^{-\frac{4\pi R}{\omega L}} \right)$$

where τ is the time at the beginning of the interval. Hence show that the differential equation only possesses an appropriate real solution when $2S^2 a^2 \geq E_0^2$. Show also that when τ is very large, $2S^2 a^2$ is approximately equal to E_0^2 , and is thus practically independent of the initial conditions.

4. Prove that equation (1) may be solved by writing $y = uw$, and determining w so that the coefficient of u vanishes.

5. Prove that the values of x for which a solution of equation (1) can become infinite or indeterminate are independent of the supplementary conditions.

6. Prove that equation (1) can only possess one solution which is a continuous function of x and has a given value when $x = 0$, except in the case when v is infinite or indeterminate for $x = 0$.

7. In the differential equation

$$\frac{dx}{dt} + \lambda x f(t) = 0$$

the constant λ is unknown. Prove that it may be determined uniquely if it is known that at time τ , x has just half its initial value when $t = 0$.

8. Show that equation (1) is transformed into another equation of the same type by the substitutions

$$x = F(\xi) \quad y = \eta G(\xi) + H(\xi)$$

where F , G , H are arbitrary functions, and η , ξ are regarded as the new dependent and independent variables respectively.

9. Solve the equations $x \frac{dy}{dx} - y = x^n$

$$\frac{dy}{dx} - xy = x$$

10. A radioactive substance A_1 is changing into another radioactive substance A_2 , while A_2 is changing into a stable substance A_3 . If P_1, P_2, P_3 are the amounts of the substances present at time t , the following differential equations are satisfied :

$$\frac{dP_1}{dt} = -\lambda_1 P_1$$

$$\frac{dP_2}{dt} = \lambda_1 P_1 - \lambda_2 P_2$$

$$\frac{dP_3}{dt} = \lambda_2 P_2$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants peculiar to the substances A_1, A_2, A_3 respectively.

If when $t = 0, P_1 = N, P_2 = 0, P_3 = 0$, the amounts of the different substances present at time t are given by the equations

$$P_1 = Ne^{-\lambda_1 t}$$

$$P_2 = \frac{N\lambda_1}{\lambda_2 - \lambda_1} [e^{-\lambda_1 t} - e^{-\lambda_2 t}]$$

$$P_3 = \frac{N}{\lambda_2 - \lambda_1} [-\lambda_2 e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_2 t} + \lambda_2 - \lambda_1]$$

§ 8. *Equations reducible to Linear Equations.* A given differential equation may sometimes be transformed into a linear equation by a change of variables. For instance, in the case of the equation *

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x only, a suitable change of variables is

$$y^{1-n} = v \quad (1 - n)y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

The transformed equation is

$$\frac{dv}{dx} + (1 - n)Pv = (1 - n)Q$$

and the method of § 7 may be used to obtain the solution.

The equation
$$\frac{dy}{dx} + Pe^y = Q$$

may be reduced to the linear form by putting

$$z = e^{-y}$$

The equation for z is then

$$\frac{dz}{dx} + Qz = P$$

* This is sometimes called Bernoulli's equation.

EXAMPLES.

1. A particle describes a curve in a vertical plane under the action of gravity, and meets a resistance which acts in a direction opposite to that in which the particle is moving at each instant, and varies as some function $f(v)$ of the particle's velocity v at this instant. Prove that the path of the particle is determined by the equations

$$\frac{1}{v} \frac{dv}{d\psi} + \cot \psi + \frac{f(v)}{g \sin \psi} = 0$$

$$gt = \int \frac{v d\psi}{\sin \psi}$$

where g is the acceleration due to gravity and ψ is the angle which the tangent to the path makes with the vertical. Integrate the first equation in the case when $f(v) = kv^n$, k being a positive constant.

2. Solve the equations $\frac{dy}{dx} + xy = y^2$

$$\frac{dy}{dx} + y \tan x = y^2$$

§ 9. *Integrating factors of two linear equations.* Let us consider the two linear equations.

$$\left. \begin{aligned} \frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= bx + ay \end{aligned} \right\} \dots \dots \dots (1)$$

where a and b are functions of t . Multiplying the first equation by $2x$, the second by $2y$ and adding, we get

$$\frac{d}{dt}(x^2 + y^2) = 2a(x^2 + y^2) \dots \dots \dots (2)$$

Hence $x^2 + y^2 = A^2 \exp \left(\int 2a dt \right)$

where A is an arbitrary constant.

Again, if we multiply the second equation by x , the first equation by y and subtract, we get

$$x \frac{dy}{dt} - y \frac{dx}{dt} = b(x^2 + y^2)$$

or
$$\frac{d}{dt} \left(\tan^{-1} \frac{y}{x} \right) = b$$

Hence
$$\tan^{-1} \frac{y}{x} = \int b dt + \epsilon \dots \dots \dots (3)$$

where ϵ is an arbitrary constant. Solving equations (2) and (3) for x and y , we find that

$$\begin{aligned} x &= Ae^{\int a dt} \cos \left(\int b dt + \epsilon \right) \\ y &= Ae^{\int a dt} \sin \left(\int b dt + \epsilon \right) \end{aligned} \quad (4)$$

If $a = 0$ and b is a constant, the solution is simply

$$x = A \cos (bt + \epsilon) \quad y = A \sin (bt + \epsilon) \quad (5)$$

Let us now consider the equations

$$\left. \begin{aligned} \frac{dx}{dt} + ax + by &= f \\ \frac{dy}{dt} - bx + ay &= g \end{aligned} \right\} \quad (6)$$

where a, b, f, g are functions of t . To solve these equations, we shall endeavour to find two functions x_0, y_0 , such that the equations

$$\left. \begin{aligned} \frac{d}{dt}(xx_0 + yy_0) &= fx_0 + gy_0 \\ \frac{d}{dt}(xy_0 - yx_0) &= fy_0 - gx_0 \end{aligned} \right\} \quad (7)$$

are a direct consequence of (6). Substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}$ given by (6) in (7), we find that the last pair of equations are satisfied identically if

$$\begin{aligned} \frac{dx_0}{dt} &= ax_0 - by_0 \\ \frac{dy_0}{dt} &= bx_0 + ay_0 \end{aligned}$$

Solving these equations by the method already explained and choosing special values of the arbitrary constants, we may write

$$x_0 = e^{\int a dt} \cos \left(\int b dt \right) \quad y_0 = e^{\int a dt} \sin \left(\int b dt \right)$$

Substituting in equations (7) and integrating, we obtain the values of x and y . If we use the notation

$$x_0^* = e^{-\int a dt} \cos \left(\int b dt \right) \quad y_0^* = e^{-\int a dt} \sin \left(\int b dt \right)$$

the complete solution may be written in the form

$$\begin{aligned} x &= x_0^* \left[\int (fx_0 + gy_0) dt + A \right] + y_0^* \left[\int (fy_0 - gx_0) dt + B \right] \\ y &= y_0^* \left[\int (fx_0 + gy_0) dt + A \right] - x_0^* \left[\int (fy_0 - gx_0) dt + B \right] \end{aligned}$$

where A and B are arbitrary constants.

EXAMPLES.

1. A charged particle starts from rest in the plane $x = 0$ at time $t = 0$, and is acted upon by a uniform electric field of strength X , parallel to the axis of x , and by a uniform magnetic force H , parallel to the axis of z . Assuming that the equations of motion are

$$m \frac{d^2x}{dt^2} = eX - He \frac{dy}{dt}$$

$$m \frac{d^2y}{dt^2} = eH \frac{dx}{dt}$$

where the mass m and the electric charge e are constants, prove that the position of the particle at time t is given by the equations

$$x = \frac{m}{e} \frac{X}{H^2} \left\{ 1 - \cos \left(\frac{e}{m} Ht \right) \right\}$$

$$y = \frac{m}{e} \frac{X}{H^2} \left\{ \frac{e}{m} Ht - \sin \left(\frac{e}{m} Ht \right) \right\}$$

2. Solve the equations given in the last example for the case in which

$$X = a \cos \omega t \quad H = b \cos \omega t$$

and a, b, ω are constants.

3. A rigid lamina moves in its own plane so that its angular velocity ω and the coordinates (ξ, η) of the centre of instantaneous rotation are given functions of t . Write down and solve the differential equations satisfied by the coordinates of an arbitrary point of the lamina.

4. Solve equations (1) and (6) by writing $x + iy = z$, where $i = \sqrt{-1}$.

5. Solve the difference equations

$$x_{n+1} = a_n x_n - b_n y_n \quad y_{n+1} = b_n x_n + a_n y_n$$

where a_n, b_n are given functions of n .

§ 10. *Systems of linear equations with constant coefficients.* Let us consider the equations

$$\left. \begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= lx + my \end{aligned} \right\} \dots \dots \dots (1)$$

where a, b, l, m are constants. If we multiply the second equation by an arbitrary constant λ and add to the first, we obtain

$$\frac{d}{dt}(x + \lambda y) = (a + \lambda l)x + (b + \lambda m)y \dots \dots (2)$$

Now choose λ so that

$$b + \lambda m = \lambda(a + \lambda l) \dots \dots \dots (3)$$

and write $z = x + \lambda y$. We then have the following equation for z :

$$\frac{dz}{dt} = (a + l\lambda)z \quad \dots \quad (4)$$

This gives us

$$z = Ce^{(a+l\lambda)t}$$

where C is an arbitrary constant.

There are generally two distinct values of λ for which equation (3) is satisfied, let us call these λ_1 and λ_2 ; then we have

$$x + \lambda_1 y = C_1 e^{(a+l\lambda_1)t} \quad x + \lambda_2 y = C_2 e^{(a+l\lambda_2)t}$$

where C_1 and C_2 are arbitrary constants. If $x = x_0$, $y = y_0$ when $t = 0$, we have $C_1 = x_0 + \lambda_1 y_0$, $C_2 = x_0 + \lambda_2 y_0$. The values of x and y are now completely determined, and we have

$$\left. \begin{aligned} x(\lambda_2 - \lambda_1) &= \lambda_2(x_0 + \lambda_1 y_0)e^{(a+l\lambda_1)t} - \lambda_1(x_0 + \lambda_2 y_0)e^{(a+l\lambda_2)t} \\ y(\lambda_1 - \lambda_2) &= (x_0 + \lambda_1 y_0)e^{(a+l\lambda_1)t} - (x_0 + \lambda_2 y_0)e^{(a+l\lambda_2)t} \end{aligned} \right\} (5)$$

If the equation for λ has two equal roots, we use the equation

$$x + \lambda y = Ce^{(a+l\lambda)t}$$

to simplify the second of equations (1). We then find that y satisfies the equation

$$\frac{dy}{dt} = (m - l\lambda)y + lCe^{(a+l\lambda)t}$$

Now, since equation (3) has two equal roots, its derivative with regard to λ must vanish; hence

$$m = a + 2l\lambda$$

The preceding equation may consequently be written in the form

$$\frac{dy}{dt} = (a + l\lambda)y + lCe^{(a+l\lambda)t}$$

or

$$\frac{d}{dt} [ye^{-(a+l\lambda)t}] = lC$$

Hence

$$ye^{-(a+l\lambda)t} = lCt + A$$

where A is an arbitrary constant.

If $x = x_0$, $y = y_0$ when $t = 0$, we have

$$C = x_0 + \lambda y_0 \quad A = y_0$$

Hence, finally,

$$\left. \begin{aligned} y &= [y_0 + l(x_0 + \lambda y_0)t] e^{(a+l\lambda)t} \\ x &= [x_0 - \lambda l(x_0 + \lambda y_0)t] e^{(a+l\lambda)t} \end{aligned} \right\} \quad \dots \quad (6)$$

If the roots of equation (3) are not real, that is, if

$$(m - a)^2 + 4bl < 0$$

we may proceed as follows. Write

$$y = \kappa v \quad x + \xi y = u$$

where κ and ξ are constants.

The differential equations satisfied by u and v are then

$$\frac{du}{dt} = (a + \xi l)u + \kappa(b + \xi m - \xi a - \xi^2 l)v$$

$$\frac{dv}{dt} = \frac{l}{\kappa}u + (m - l\xi)v$$

We now choose ξ and κ , so that

$$a + \xi l = m - l\xi$$

$$\kappa(b + \xi m - \xi a - \xi^2 l) = -\frac{l}{\kappa}$$

the differential equations for u and v are then of the type considered in § 9. Solving for ξ and κ , we get

$$2l\xi = m - a \qquad 4\frac{l^2}{\kappa^2} = -4bl - (m - a)^2 = +4\omega^2, \text{ say.}$$

Writing $a + m = 2\sigma$, our differential equations take the forms

$$\frac{du}{dt} = \sigma u - \omega v$$

$$\frac{dv}{dt} = \omega u + \sigma v$$

and the solution is consequently

$$u = Ae^{\sigma t} \cos \omega(t - t_0) \qquad v = Ae^{\sigma t} \sin \omega(t - t_0)$$

where A and t_0 are arbitrary constants.

The corresponding expressions for x and y are now found to be

$$\left. \begin{aligned} y &= Axe^{\sigma t} \sin \omega(t - t_0) \\ x &= Ae^{\sigma t} \left[\cos \omega(t - t_0) - \frac{m - a}{2\omega} \sin \omega(t - t_0) \right] \end{aligned} \right\} \quad (7)$$

If we wish to solve the equations

$$\frac{dx}{dt} = ax + by + f(t)$$

$$\frac{dy}{dt} = lx + my + g(t)$$

we may adopt a similar method. If the roots of equation (3) are real and distinct, we write $z_1 = x + \lambda_1 y$, $z_2 = x + \lambda_2 y$, and obtain

$$\frac{dz_1}{dt} = (a + \lambda_1 l)z_1 + f(t) + \lambda_1 g(t)$$

$$\frac{dz_2}{dt} = (a + \lambda_2 l)z_2 + f(t) + \lambda_2 g(t)$$

These equations can be solved by the method of § 7, and the

values of x and y derived from the expressions obtained for z_1 and z_2 .

When the roots of equation (3) are not real, we write

$$x + \xi y = u \quad y = \kappa v$$

as before, and obtain

$$\frac{du}{dt} = \sigma u - \omega v + f(t) + \xi g(t)$$

$$\frac{dv}{dt} = \omega u + \sigma v + \frac{1}{\kappa} g(t)$$

These equations can be solved by the method of § 9.

When equation (3) has two equal roots, we write $z = x + \lambda y$, and obtain

$$\frac{dz}{dt} = (a + \lambda l)z + f(t) + \lambda g(t)$$

$$\frac{dy}{dt} = (a + \lambda l)y + lz + g(t)$$

The first equation can be solved by the method of § 7, and when the value of z so found is substituted in the second equation, this new equation can be solved by the same method.

EXAMPLES.

1. If in equations (1) $x = 0$ for $t = 0$ and $y = 0$ for $t = \tau$, the functions x and y are always zero unless

$$\lambda_1 e^{\lambda_1 \tau} = \lambda_2 e^{\lambda_2 \tau}$$

When this condition is satisfied, the value of y_0 can be chosen arbitrarily.

2. The magnitudes x and y of the electric currents flowing in two circuits acted on by electromotive forces $f(t)$, $g(t)$ respectively, satisfy the differential equations

$$L \frac{dx}{dt} + M \frac{dy}{dt} + Rx = f(t)$$

$$M \frac{dx}{dt} + N \frac{dy}{dt} + Sy = g(t)$$

where L , N are the coefficients of self-induction, R , S the resistances and M the mutual inductance of the two circuits. Integrate the equations on the supposition that L , M , N , R , S are positive constants.

3. Solve the difference equations

$$x_{n+1} = ax_n + by_n + f_n$$

$$y_{n+1} = lx_n + my_n + g_n$$

where a , b , l , m are constants and f_n , g_n given functions of the positive integer n .

§ 11. *Linear equations of the second order with constant coefficients.*

Let us consider the differential equation

$$\frac{d^2x}{dt^2} = m \frac{dx}{dt} + lx$$

in which l and m are constants. Putting $\frac{dx}{dt} = y$, we obtain the two linear equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = lx + my$$

Treating these by the methods of § 10, we see that if the equation

$$l\lambda^2 = \lambda m + 1$$

has two distinct roots λ_1, λ_2 , and we write $\alpha_1 = l\lambda_1, \alpha_2 = l\lambda_2$, the solution is given by the formula

$$lx(\alpha_2 - \alpha_1) = \alpha_2(lx_0 + \alpha_1 y_0)e^{\alpha_1 t} - \alpha_1(lx_0 + \alpha_2 y_0)e^{\alpha_2 t}$$

This is the solution corresponding to the supplementary conditions

$$x = x_0 \quad \frac{dx}{dt} = y_0 \quad \text{when } t = 0$$

It should be noticed that the general solution is of the form

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

where A_1, A_2 are arbitrary constants and α_1, α_2 are the roots of the equation

$$\alpha^2 = m\alpha + l$$

When this equation has equal roots, so that $\alpha_1 = \alpha_2 = \alpha$, we see from equation (6) of § 10 that

$$lx = [lx_0 - \alpha(lx_0 + \alpha y_0)t] e^{\alpha t}$$

When the roots of the equation for α are not real, it follows from (7) that the solution is of the form

$$x = e^{\sigma t} [B \cos \omega t + C \sin \omega t]$$

where B and C are arbitrary constants, since A and t_0 were arbitrary. The constants σ and ω are given by the equations

$$2\sigma = m \quad 4\omega^2 + m^2 + 4l = 0$$

The equation $\frac{d^2x}{dt^2} = m \frac{dx}{dt} + lx + f(t)$

may be replaced by a pair of linear equations by writing

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = lx + my + f(t)$$

and the methods of § 10 are again applicable.

To illustrate the method of obtaining the solution, let us first of all consider the equation

$$\frac{d^2x}{dt^2} - (\alpha_1 + \alpha_2) \frac{dx}{dt} + \alpha_1\alpha_2x = f(t)$$

This may be replaced by the pair of linear equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = (\alpha_1 + \alpha_2)y - \alpha_1\alpha_2x + f(t)$$

The equation for α is now

$$\alpha^2 - \alpha(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 \equiv (\alpha - \alpha_1)(\alpha - \alpha_2) = 0.$$

Writing $y - \alpha_1x = z_1$, $y - \alpha_2x = z_2$, we obtain the two linear equations

$$\frac{dz_1}{dt} = \alpha_2z_1 + f(t)$$

$$\frac{dz_2}{dt} = \alpha_1z_2 + f(t)$$

Hence $y - \alpha_1x = z_1 = A_1e^{\alpha_2t} + e^{\alpha_2t} \int e^{-\alpha_2t} f(t) dt$

$$y - \alpha_2x = z_2 = A_2e^{\alpha_1t} + e^{\alpha_1t} \int e^{-\alpha_1t} f(t) dt$$

where A_1 and A_2 are arbitrary constants. Solving for x , we find that

$$(\alpha_1 - \alpha_2)x = A_2e^{\alpha_1t} - A_1e^{\alpha_2t} + e^{\alpha_1t} \int e^{-\alpha_1t} f(t) dt - e^{\alpha_2t} \int e^{-\alpha_2t} f(t) dt$$

In this solution the part which involves the function $f(t)$ is called the *particular integral*, the term $A_2e^{\alpha_1t} - A_1e^{\alpha_2t}$ is called the *complementary function*.

When $\alpha_1 = \alpha_2 = \alpha$, the quickest way of solving the equation

$$\frac{d^2x}{dt^2} - 2\alpha \frac{dx}{dt} + \alpha^2x = f(t)$$

is to make use of the fact that $e^{-\alpha t}$ and $te^{-\alpha t}$ are integrating factors; we thus obtain

$$\frac{d}{dt} \left[e^{-\alpha t} \frac{dx}{dt} - \alpha e^{-\alpha t} x \right] = e^{-\alpha t} f(t)$$

$$\frac{d}{dt} \left[te^{-\alpha t} \frac{dx}{dt} - \alpha \left(t + \frac{1}{\alpha} \right) e^{-\alpha t} x \right] = te^{-\alpha t} f(t)$$

whence $e^{-\alpha t} \frac{dx}{dt} - \alpha x e^{-\alpha t} = A_1 + \int e^{-\alpha \tau} f(\tau) d\tau$

$$te^{-\alpha t} \frac{dx}{dt} - (\alpha t + 1) x e^{-\alpha t} = A_2 + \int e^{-\alpha \tau} \tau f(\tau) d\tau$$

Solving for x , we get

$$x = e^{at}(A_1 t - A_2) + e^{at} \int e^{-a\tau}(t - \tau)f(\tau) d\tau$$

where A_1 and A_2 are arbitrary constants.

In particular, if $f(t) = Ae^{at}$ where A is a constant, we find that

$$x = e^{at}(A_1 t - A_2 + \frac{1}{2}At^2)$$

Let us next consider the equation of forced damped vibrations

$$\frac{d^2x}{dt^2} + \kappa \frac{dx}{dt} + \mu^2 x = F(t)$$

where $\kappa \frac{dx}{dt}$ represents a frictional force proportional to the velocity and resisting the motion, $\mu^2 x$ the spring or force of restitution and $F(t)$ the generating force. We shall consider the case when $\kappa = 2a$, and $\mu^2 = a^2 + b^2$ where a and b are positive constants. The differential equation is then

$$\frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + (a^2 + b^2)x = F(t)$$

If now we define y by the equation

$$\frac{dx}{dt} + ax + by = 0$$

we find that $\frac{dy}{dt} - bx + ay = -\frac{1}{b}F(t)$

Comparing the last two equations with equations (6) of § 9, we may conclude that the functions

$$x_0 = e^{at} \cos bt, \quad y_0 = e^{at} \sin bt$$

are integrating factors. Hence

$$xx_0 + yy_0 = -\frac{1}{b} \int_0^t e^{a\tau} \sin b\tau F(\tau) d\tau + A$$

$$yx_0 - xy_0 = -\frac{1}{b} \int_0^t e^{a\tau} \cos b\tau F(\tau) d\tau + B$$

where A and B are arbitrary constants.

Solving for x , we finally obtain

$$x = \frac{1}{b} \int_0^t e^{a(\tau-t)} \sin b(t-\tau) F(\tau) d\tau + Ae^{-at} \cos bt - Be^{-at} \sin bt$$

If the supplementary conditions state that $x = 0$, $\frac{dx}{dt} = 0$ when $t = 0$, we have $y = 0$ when $t = 0$, and it follows that $A = B = 0$; hence the solution is in this case

$$x = \frac{1}{b} \int_0^t e^{a(\tau-t)} \sin b(t-\tau) F(\tau) d\tau$$

Let us now consider the special case in which $F(t)$ is a particular solution of the equation

$$\frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + (a^2 + b^2)x = 0$$

We may then write

$$F(t) = Ee^{-at} \sin(bt + \epsilon)$$

where E and ϵ are constants. The general formula for x then gives

$$x = \frac{E}{b} e^{-at} \int_0^t \sin b(t - \tau) \sin(b\tau + \epsilon) d\tau + Ae^{-at} \cos bt - Be^{-at} \sin bt$$

Now

$$2 \sin b(t - \tau) \sin(b\tau + \epsilon) = \cos(bt - 2b\tau - \epsilon) - \cos(bt + \epsilon)$$

hence it appears that

$$x = -\frac{Et}{2b} e^{-at} \cos(bt + \epsilon) + A_1 e^{-at} \cos bt - B_1 e^{-at} \sin bt$$

where A_1, B_1 are arbitrary constants differing in value from A and B .

If $a = 0$, so that the differential equation reduces to

$$\frac{d^2x}{dt^2} + b^2x = E \sin(bt + \epsilon)$$

we have
$$x = -\frac{Et}{2b} \cos(bt + \epsilon) + A_1 \cos bt - B_1 \sin bt$$

The presence of the term $-\frac{Et}{2b} \cos(bt + \epsilon)$ indicates that the effect of the force $E \sin(bt + \epsilon)$ becomes more and more pronounced as t increases. In this case the period of the force $F(t)$ is the same as the period of the free motion when $F(t) = 0$. Let us next consider the case in which the period of the force differs from that of the free motion, but the damping coefficient a is the same. In other words, let

$$F(t) = Ee^{-at} \sin(pt + \epsilon)$$

where E, p, ϵ are constants. We then have to calculate the integral

$$I = \int_0^t \sin b(t - \tau) \sin(p\tau + \epsilon) d\tau$$

Now
$$2 \sin b(t - \tau) \sin(p\tau + \epsilon)$$

$$= \cos[bt - (b + p)\tau - \epsilon] - \cos[bt - (b - p)\tau + \epsilon]$$

Hence
$$2I = \frac{1}{b + p} [\sin(bt - \epsilon) + \sin(pt + \epsilon)]$$

$$+ \frac{1}{b - p} [\sin(pt + \epsilon) - \sin(bt + \epsilon)]$$

i.e.
$$I = \frac{b}{b^2 - p^2} \sin(pt + \epsilon) - \frac{p \sin bt \cos \epsilon + b \cos bt \sin \epsilon}{b^2 - p^2}$$

Substituting in the formula

$$x = \frac{E}{b} e^{-at} \int_0^t \sin b(t - \tau) \sin(p\tau + \varepsilon) d\tau + A e^{-at} \cos bt - B e^{-at} \sin bt$$

we find that

$$x = \frac{E}{b^2 - p^2} e^{-at} \sin(pt + \varepsilon) + A_1 e^{-at} \cos bt + B_1 e^{-at} \sin bt$$

where A_1 and B_1 are arbitrary constants.

EXAMPLES.

1. Solve the equations

$$\frac{d^2x}{dt^2} - 5 \frac{dx}{dt} + 6x = e^{2t}$$

$$\frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 2x = e^{-t} \cos t$$

$$\frac{d^2x}{dt^2} + b^2x = \sin^2 \frac{1}{2} bt$$

$$\frac{d^2x}{dt^2} - 4 \frac{dx}{dt} + 4x = t + e^{2t}$$

2. Solve the equations $\frac{d^2x}{dt^2} = ax + by$

$$\frac{d^2y}{dt^2} = lx + my$$

on the supposition that a, b, l, m are constants, and that

$$(m - a)^2 + 4bl > 0$$

3. Solve the equation $\frac{d^2x}{dt^2} - 2\alpha \frac{dx}{dt} + \alpha^2 x = f(t)$
by putting $x = e^{\alpha t} y$.

4. Solve the difference equation

$$x_{n+2} + ax_{n+1} + bx_n = 0$$

by putting $x_{n+1} = y_n$. Hence show that the solution can be expressed in the form

$$x_n = A_1 \lambda_1^n + A_2 \lambda_2^n$$

where λ_1 and λ_2 are the roots of the equation $\lambda^2 + a\lambda + b = 0$. Determine the form of the solution when $a^2 = 4b$.

5. Prove that the equation

$$y_{n+2} - 2\alpha \cos \beta y_{n+1} + \alpha^2 y_n = 0$$

is satisfied by

$$y_n = A \alpha^n \cos(n\beta + \varepsilon)$$

where A and ε are arbitrary constants.

6. Solve the difference equation

$$y_{n+2} + ay_{n+1} + by_n = f_n.$$

§ 12. *Symbolical methods.* Most of the results of the preceding article may be obtained by a symbolical method in which the operator $D \equiv \frac{d}{dt}$ is treated as if it satisfied the ordinary laws of Algebra. For instance, the equation

$$\frac{d^2x}{dt^2} - (\alpha_1 + \alpha_2) \frac{dx}{dt} + \alpha_1 \alpha_2 x = f(t) \quad (1)$$

may be written in the form

$$(D - \alpha_1)(D - \alpha_2)x = f(t) \quad (2)$$

To solve the equation, we must find a value for the expression

$$\frac{1}{(D - \alpha_1)(D - \alpha_2)} f(t) \quad (3)$$

Resolving the operator into partial fractions, we replace the foregoing expression by

$$x = \frac{1}{\alpha_2 - \alpha_1} \cdot \frac{1}{D - \alpha_1} f(t) - \frac{1}{\alpha_2 - \alpha_1} \cdot \frac{1}{D - \alpha_2} f(t) \quad (4)$$

Now the expression $(D - \alpha_1)^{-1} f(t)$ must be regarded as the solution of the equation

$$(D - \alpha_1)x_1 = f(t)$$

and has consequently the value

$$x_1 = A_1 e^{\alpha_1 t} + e^{\alpha_1 t} \int e^{-\alpha_1 t} f(t) dt$$

where A_1 is an arbitrary constant. Evaluating the second term in (4) in a similar way, we finally obtain

$$(\alpha_2 - \alpha_1)x = A_1 e^{\alpha_1 t} - A_2 e^{\alpha_2 t} + e^{\alpha_1 t} \int e^{-\alpha_1 t} f(t) dt - e^{\alpha_2 t} \int e^{-\alpha_2 t} f(t) dt$$

This agrees with our former expression; consequently the symbolical method gives the correct result.

In using the symbolical method, the following rules are sometimes useful :

$$F(D) \cdot e^{\alpha t} f(t) = e^{\alpha t} F(D + \alpha) f(t)$$

$$F(D^2) \sin(pt + \epsilon) = F(-p^2) \sin(pt + \epsilon)$$

When the function F is a rational integral function of its argument the preceding equations have an intelligible meaning. It is easy to verify the equations for the case in which $F(D) = D^n$, n being a positive integer; consequently we may conclude that the equations hold when $F(D)$ is a polynomial in D . To extend the equations to the case in which $F(D)$ is the ratio of two polynomials in D , we must understand the equations to mean that the right-hand side is one value of the operation on the left-hand side. When $f(t)$ is a polynomial a value of $F(D)f(t)$ may be calculated by expanding $F(D)$ in ascending powers of D and allowing each term to operate

on $f(t)$. After a certain term the result of each operation is zero, and so $F(D)f(t)$ is obtained as the sum of a finite number of terms.

As a further illustration of the symbolical method, let us take the equation

$$\frac{d^2x}{dt^2} + b^2x = E \sin(pt + \epsilon)$$

A particular integral is given by

$$\begin{aligned} x &= \frac{1}{D^2 + b^2} E \sin(pt + \epsilon) \\ &= \frac{E}{b^2 - p^2} \sin(pt + \epsilon) \end{aligned}$$

To obtain the complete solution, we must add the *complementary function*, that is the general solution of the equation

$$\frac{d^2x}{dt^2} + b^2x = 0$$

$$\text{Hence } x = \frac{E}{b^2 - p^2} \sin(pt + \epsilon) + A \cos bt + B \sin bt$$

where A and B are arbitrary constants. This agrees with a result obtained by a former method. If $p = b$, the symbolical method requires modification.*

The symbolical method gives us a rapid method of evaluating certain integrals. For instance, if we wish to integrate $e^{at} \sin pt$, we write

$$\begin{aligned} \frac{1}{D} e^{at} \sin pt &= e^{at} \frac{1}{D + \alpha} \sin pt = e^{at} \frac{D - \alpha}{D^2 - \alpha^2} \sin pt \\ &= e^{at} \frac{\alpha - D}{\alpha^2 + p^2} \sin pt = e^{at} \frac{\alpha \sin pt - p \cos pt}{\alpha^2 + p^2} \end{aligned}$$

It should be noticed that in the preceding analysis we have replaced D^2 by $-p^2$ in the operator acting on $\sin pt$. This is an application of the following rule, which is justified by the fact that it gives correct results:

Rule. In calculating a particular value of $F(D) \sin(pt + \epsilon)$, we may simplify the operator by replacing D^2 by $-p^2$ wherever an even power of D occurs.

Let us now find a particular integral of the equation of forced oscillations

$$\frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + \mu^2x = Ee^{-\lambda t} \sin(pt + \epsilon)$$

* The correct result may be obtained by finding the limiting value of the above expression for x as $p \rightarrow b$. If we differentiate the numerator and denominator of the fraction $\frac{E \sin(pt + \epsilon)}{b^2 - p^2}$ with regard to p and then put $p = b$, we obtain the limiting value

$$-\frac{Et}{2b} \cos(bt + \epsilon)$$

wherein $a, \mu, E, \lambda, p, \epsilon$ are real constants. We write

$$\begin{aligned} x &= \frac{E}{D^2 + 2aD + \mu^2} e^{-\lambda t} \sin(pt + \epsilon) \\ &= Ee^{-\lambda t} \frac{1}{(D - \lambda)^2 + 2a(D - \lambda) + \mu^2} \sin(pt + \epsilon) \\ &= Ee^{-\lambda t} \frac{1}{\lambda^2 - 2a\lambda + \mu^2 - p^2 + 2D(a - \lambda)} \sin(pt + \epsilon) \\ &= Ee^{-\lambda t} \frac{\lambda^2 - 2a\lambda + \mu^2 - p^2 - 2D(a - \lambda)}{(\lambda^2 - 2a\lambda + \mu^2 - p^2)^2 - 4D^2(a - \lambda)^2} \sin(pt + \epsilon) \\ &= Ee^{-\lambda t} \frac{(\lambda^2 - 2a\lambda + \mu^2 - p^2) \sin(pt + \epsilon) - 2p(a - \lambda) \cos(pt + \epsilon)}{(\lambda^2 - 2a\lambda + \mu^2 - p^2)^2 + 4p^2(a - \lambda)^2} \end{aligned}$$

To obtain the complete solution, we must add the complementary function, which, if $\mu^2 > a^2$, has the form

$$Ae^{-at} \cos bt + Be^{-at} \sin bt$$

where $\mu^2 = a^2 + b^2$.

The case in which $\lambda = 0, p = \mu, a > 0$ is of special interest, for then the solution is

$$x = -\frac{E}{2a\mu} \cos(\mu t + \epsilon) + Ae^{-at} \cos bt + Be^{-at} \sin bt$$

where A and B are constants which depend on the initial conditions. As $t \rightarrow \infty$, the last two terms become negligible and the amplitude of the forced oscillation depends ultimately on the quantity $\frac{E}{2a\mu}$, which is large when the damping coefficient a is small. It should be noticed that the amplitude varies directly as the period of the impressed force.

To solve the equation $(D - \alpha)^2 x = f(t)$ by the symbolical method, we write

$$\begin{aligned} x &= \frac{1}{(D - \alpha)^2} f(t) \\ &= \frac{1}{(D - \alpha)^2} e^{at} [e^{-at} f(t)] \\ &= e^{at} \frac{1}{D^2} [e^{-at} f(t)] \\ &= e^{at} \left[A + Bt + \int^t (t - \tau) e^{-a\tau} f(\tau) d\tau \right] \end{aligned}$$

The symbolical method may be used with advantage in the solution of systems of linear equations with constant coefficients. Let us consider the following example in the theory of electric currents. Currents are generated in two coupled circuits by a

variable electromotive force $f(t)$ acting on the first circuit. If L, N are the coefficients of self-induction of the two circuits, R, S the resistances and M the coefficient of self-induction, the currents x and y in the two circuits satisfy the differential equations

$$L \frac{dx}{dt} + M \frac{dy}{dt} + Rx = f(t)$$

$$M \frac{dx}{dt} + N \frac{dy}{dt} + Sy = 0$$

Writing these in the forms

$$(LD + R)x + MDy = f(t)$$

$$MDx + (ND + S)y = 0$$

we may satisfy the last equation by putting

$$x = (ND + S)u \quad y = -MDu$$

Substituting in the first equation, we get

$$[(LD + R)(ND + S) - M^2D^2]u = f(t)$$

Since $(RN + SL)^2 - 4RS(LN - M)^2 \equiv (RN - SL)^2 + 4RSM^2$

and is positive, the complementary function in the solution of the last equation is of the form $Ae^{-\alpha t} + Be^{-\beta t}$, where α and β are real and positive if, as in the actual case, $LN > M^2$. When u has been found, x and y may be derived at once by the equations given above.

The symbolical method may also be used to solve a differential equation of the n^{th} order with constant coefficients. Let us suppose that the equation is represented symbolically by

$$(D - \alpha_1)(D - \alpha_2) \dots (D - \alpha_n)x = f(t)$$

If the constants α_r are all real and distinct, we may write

$$\frac{1}{(D - \alpha_1) \dots (D - \alpha_n)} = \sum_{m=1}^n \frac{\beta_m}{D - \alpha_m}$$

the coefficient β_m being determined by the method of partial fractions.

Writing
$$x_m = \frac{1}{D - \alpha_m} f(t)$$

we have

$$x = \sum_{m=1}^n \beta_m x_m$$

The quantities x_m are obtained by solving the first-order equations

$$(D - \alpha_m)x_m = f(t).$$

In the general case a linear equation of the n^{th} order with constant coefficients will have the form

$$F(D)x = f(t) \quad \dots \quad (5)$$

where $F(D)$ is a polynomial in D , whose roots are not necessarily all real and distinct. Let us assume that

$$F(D) = (D - \alpha_1)^{\lambda_1} (D - \alpha_2)^{\lambda_2} \dots [(D - \xi)^2 + \eta^2] \dots \quad (6)$$

where the quantities $\alpha_1, \lambda_1, \alpha_2, \lambda_2, \dots, \xi, \eta$ are all real.

When we express $\frac{1}{F(D)}$ in partial fractions, we have terms of types

$$\frac{1}{(D - \alpha_1)^\nu} \quad \frac{mD + n}{(D - \xi)^2 + \eta^2}$$

hence, if we wish to calculate $\frac{1}{F(D)} f(t)$, we must find the values of

$$\frac{mD + n}{(D - \xi)^2 + \eta^2} f(t)$$

and

$$\frac{1}{(D - \alpha_1)^\nu} f(t)$$

The former may be regarded as the solution of the equation

$$[(D - \xi)^2 + \eta^2] u = f(t)$$

and has already been found; the latter may be replaced by

$$e^{\alpha_1 t} \cdot \frac{1}{D^\nu} [e^{-\alpha_1 t} f(t)]$$

and may be calculated by direct integration.

When a particular integral of equation (5) has been found, the complementary function must be added to it. The complementary function may be obtained by adding together the complementary functions corresponding to the different factors on the right-hand side of (6). It is, in fact,

$$e^{\alpha_1 t} [A_0 + A_1 t + \dots A_{\lambda_1 - 1} t^{\lambda_1 - 1}] + \dots + e^{\xi t} [P \cos \eta t + Q \sin \eta t] + \dots$$

where $A_0 \dots A_{\lambda_1 - 1}$, P , $Q \dots$ are arbitrary constants.

The case in which a factor of type $[(D - \xi)^2 + \eta^2]^e$ appears in $F(D)$ is rather difficult, and will not be discussed here. Perhaps the simplest method is to resolve it into complex factors and work with complex quantities: the preceding method may then be used.

For a fuller account of the symbolical method, the reader is referred to Forsyth's *Differential Equations*.

EXAMPLES.

1. Solve the equations

$$\frac{d^4 x}{dt^4} + 4x = t^2 + \sin t$$

$$\frac{d^4 y}{dx^4} + 3 \frac{d^2 y}{dx^2} + 2y = x^2 \cos x$$

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 4 = e^x + 2e^{4x}$$

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x^3 + e^x \sin x$$

$$\frac{d^3 x}{dt^3} - x = e^t$$

2. Show that the equation

$$\frac{d^4 y}{dx^4} + 2n^2 \frac{d^2 y}{dx^2} + n^4 = f(x)$$

may be replaced by two equations

$$\frac{d^2 z}{dx^2} + n^2 z = f(x)$$

$$\frac{d^2 y}{dx^2} + n^2 y = z$$

Hence solve the equation.

3. Show that the preceding method can be generalised so as to be applicable to the equation

$$[(D - \alpha)^2 + \beta^2]^n x = f(x)$$

4. The free electrical vibrations in two coupled circuits of negligible resistance each containing a capacity are determined by equations of type

$$L \frac{d^2 \xi}{dt^2} + M \frac{d^2 \eta}{dt^2} + \frac{1}{C} \xi = 0 \quad M \frac{d^2 \xi}{dt^2} + N \frac{d^2 \eta}{dt^2} + \frac{1}{C'} \eta = 0$$

where the capacities C , C' and the coefficients of induction L , M , N are positive constants. Prove that the vibrations are undamped.

(The currents are represented by the quantities $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$.)

§ 13. *Discontinuous solutions of linear equations with constant coefficients.* There is an interesting kind of maintained vibration in which the force of restitution or "spring" of the body susceptible of vibration is subject to an imposed periodic variation.* If this variation takes place continuously, the differential equation giving the motion may have a simple form, such as

$$\frac{d^2 x}{dt^2} + \kappa \frac{dx}{dt} + (\mu^2 - 2\alpha \sin 2pt)x = 0 \quad \dots \quad (1)$$

where the term $-2\alpha \sin 2pt$ indicates the manner in which the spring is changed. Such an equation is, however, by no means easy to solve. To obtain a case in which the integration can be effected without much trouble, we shall suppose that there is no friction, so that $\kappa = 0$, and shall assume that the periodic changes in the spring are of an impulsive nature, the spring having its normal value except during very short intervals, which occur periodically at times τ , 2τ , 3τ , etc.

Let us assume then that the equation of motion is

$$\frac{d^2 x}{dt^2} + \mu^2 x = 0, \quad \text{for } 0 < t < \tau - \epsilon, \tau + \epsilon < t < 2\tau - \epsilon, \text{ etc.}$$

$$\text{and } \frac{d^2 x}{dt^2} + (\mu^2 + \xi)x = 0, \quad \text{for } \tau - \epsilon < t < \tau + \epsilon$$

where ξ is large and ϵ small.

* See papers by Lord Rayleigh, *Phil. Mag.*, Vol. 24 (1887), *Scientific Papers*, Vol. 3, and A. Stephenson, *Quarterly Journal* (1906); *Phil. Mag.*, Vol. 14 (1907), *Ibid.* Feb. and Oct. (1908), *Manchester Memoirs*, Vol. 52 (1908).

Integrating the last equation from $\tau - \varepsilon$ to $\tau + \varepsilon$ and making ε tend to zero, we find that

$$\left[\frac{dx}{dt} \right] + \lim_{\varepsilon \rightarrow 0} \int_{\tau-\varepsilon}^{\tau+\varepsilon} \xi x dt = 0$$

where $\left[\frac{dx}{dt} \right]$ denotes the discontinuity in the value of $\frac{dx}{dt}$ at $t = \tau$.

We shall assume that the second term in the preceding equation may be replaced by $\lambda x(\tau)$, where λ is a constant and $x(\tau)$ denotes the value of x when $t = \tau$; we shall also assume that x is continuous in passing through the value τ .

If the impulses at times $2\tau, 3\tau$, etc., are similar in nature to that at time τ , the problem to be solved may be enunciated as follows:

We require a solution of the differential equation

$$\frac{d^2x}{dt^2} + \mu^2 x = 0$$

and the supplementary conditions

$$[x] = 0 \quad \left[\frac{dx}{dt} \right] + \lambda x = 0, \quad \text{for } t = \tau, 2\tau, 3\tau, \text{ etc.}$$

We see from the differential equation that we must have

$$x = A_1 \cos \mu t + B_1 \sin \mu t, \quad \text{for } t < \tau$$

$$x = A_2 \cos \mu t + B_2 \sin \mu t, \quad \text{for } t > \tau$$

where the A's and B's are constants. Since x is continuous, we must have $(A_2 - A_1) \cos \mu \tau + (B_2 - B_1) \sin \mu \tau = 0$

and since $\left[\frac{dx}{dt} \right] + \lambda x = 0$ when $t = \tau$, we must have

$$- \mu(A_2 - A_1) \sin \mu \tau + \mu(B_2 - B_1) \cos \mu \tau + \lambda(A_1 \cos \mu \tau + B_1 \sin \mu \tau) = 0$$

Solving for A_2 and B_2 , we find that

$$A_2 = A_1 + \frac{\lambda}{\mu} \sin \mu \tau (A_1 \cos \mu \tau + B_1 \sin \mu \tau)$$

$$B_2 = B_1 - \frac{\lambda}{\mu} \cos \mu \tau (A_1 \cos \mu \tau + B_1 \sin \mu \tau)$$

The values of A_1 and B_1 must be determined from the initial conditions, and when these are known the above equations will determine A_2 and B_2 .

If A_n, B_n denote the values of the constants just before the n^{th} impulse, we find in a similar way that

$$A_{n+1} = A_n + \frac{\lambda}{\mu} \sin n\mu\tau (A_n \cos n\mu\tau + B_n \sin n\mu\tau)$$

$$B_{n+1} = B_n - \frac{\lambda}{\mu} \cos n\mu\tau (A_n \cos n\mu\tau + B_n \sin n\mu\tau)$$

If τ is half the period of a free vibration under the normal spring, we have $\mu\tau = \pi$, and consequently $\sin n\mu\tau = 0$. The preceding equations now become

$$A_{n+1} = A_n \quad B_{n+1} = B_n - \frac{\lambda}{\mu} A_n$$

It follows at once that

$$A_{n+1} = A_1 \quad B_{n+1} = B_1 - \frac{n\lambda}{\mu} A_1$$

If B_1 and A_1 have opposite signs, the impulses at once magnify the amplitude of the initial vibration, and we get a vibration whose amplitude increases continually with n . If B_1 and A_1 have the same sign, the impulses may at first diminish the amplitude of the initial vibration, but as soon as $\frac{n\lambda}{\mu} A_1$ becomes greater than B_1 , the amplitude of the vibration increases continually with n . If $A_1 = 0$, the impulses do not produce any effect.

The foregoing theory throws some light upon an experiment due to Melde, in which a fine string is maintained in transverse vibration by connecting one of its extremities with the vibrating prong of a massive tuning-fork, *the direction of motion of the point of attachment being parallel to the length of the string*. The other end of the string is fixed. The effect of the motion is to render the tension of the string periodically variable; and at first sight there is nothing to cause the string to depart from its equilibrium condition of straightness. It is known, however, that under these circumstances the equilibrium may become unstable, and that the string may settle down into a state of permanent and vigorous vibration, whose period is *double* that of the fork (compare Rayleigh's *Sound*, Vol. I. p. 82).

If τ is any integral multiple of a half period, the theory is exactly the same. This corresponds to a result obtained by A. Stephenson in a discussion of equation (1). A complete discussion of the case in which τ is arbitrary is difficult; the case in which τ is a quarter period is simpler and will now be discussed.

We now have $\mu\tau = \frac{\pi}{2}$, and the equations take the form

$$\begin{aligned} A_{2n+1} &= A_{2n} & B_{2n+1} &= B_{2n} - \frac{\lambda}{\mu} A_{2n} \\ A_{2n+2} &= A_{2n+1} + \frac{\lambda}{\mu} B_{2n+1} & B_{2n+2} &= B_{2n+1} \end{aligned}$$

Hence

$$A_{2n+1} = A_{2n-1} + \frac{\lambda}{\mu} B_{2n-1} \quad B_{2n+1} = B_{2n-1} - \frac{\lambda}{\mu} A_{2n+1}$$

To solve these equations we multiply the second by ξ and add to the first; then

$$A_{2n+1}\left(1 + \frac{\xi\lambda}{\mu}\right) + \xi B_{2n+1} = A_{2n-1} + \left(\frac{\lambda}{\mu} + \xi\right)B_{2n-1}$$

Now choose ξ so that

$$\xi = \left(1 + \frac{\xi\lambda}{\mu}\right)\left(\frac{\lambda}{\mu} + \xi\right)$$

i.e. so that
$$\xi^2 + \frac{\lambda}{\mu}\xi + 1 = 0$$

If $\sigma = \frac{\lambda}{\mu} + \xi$ where ξ is a root of the preceding equation, we have $\xi\sigma + 1 = 0$ and

$$A_{2n+1} + \sigma B_{2n+1} = -\sigma^2(A_{2n-1} + \sigma B_{2n-1})$$

or
$$A_{2n+1} + \sigma B_{2n+1} = (-1)^n \sigma^{2n}(A_1 + \sigma B_1)$$

If σ_1, σ_2 are the two values of σ , we have $\sigma_1\sigma_2 = 1$; hence if σ_1 is real and numerically less than 1, σ_2 is real and numerically greater than 1. Hence, if σ_1 and σ_2 are real, at least one of them is numerically greater than 1, and the equations

$$A_{2n+1} + \sigma_1 B_{2n+1} = (-1)^n \sigma_1^{2n}(A_1 + \sigma_1 B_1)$$

$$A_{2n+1} + \sigma_2 B_{2n+1} = (-1)^n \sigma_2^{2n}(A_1 + \sigma_2 B_1)$$

indicate that the amplitude of the vibration is large when n is very large.

The two values of σ are real when $\lambda^2 > \mu^2$; consequently, if this condition is satisfied, the effect of the sequence of impulses is to produce ultimately a vibration of continually increasing amplitude. The impulses thus have a cumulative effect.

If $\lambda^2 < \mu^2$, the values of σ are of the form $e^{i\theta}, e^{-i\theta}$ respectively, where θ is real and $i = \sqrt{-1}$. In this case the values of A_{2n+1} and B_{2n+1} remain finite as n increases and fluctuate between certain upper and lower limits. The impulses do not produce a vibration of continually increasing amplitude.

§ 14. *A system of linear equations occurring in the theory of radio-active transformations.* Let us suppose that a radio-active substance A_0 is changing into A_1 , that A_1 is changing into A_2 , A_2 into A_3 , and so on. Let $P_0, P_1, \dots, P_n$ be the amounts of $A_0, A_1, \dots, A_n$ present at time t . Then, from the law of mass-action, the rate at which A_s is transformed into A_{s+1} is proportional to P_s , while the rate at which A_{s-1} is being transformed into A_s is proportional to P_{s-1} . The total rate at which the quantity P_s is changing is thus $\lambda_{s-1}P_{s-1} - \lambda_s P_s$, where the quantities λ_{s-1}, λ_s are characteristic

constants for the substances A_{s-1} , A_s respectively. We thus have the system of equations*

$$\begin{aligned}\frac{dP_0}{dt} &= -\lambda_0 P_0 \\ \frac{dP_1}{dt} &= \lambda_0 P_0 - \lambda_1 P_1 \\ \frac{dP_2}{dt} &= \lambda_1 P_1 - \lambda_2 P_2 \\ &\dots \dots \dots \\ \frac{dP_n}{dt} &= \lambda_{n-1} P_{n-1} - \lambda_n P_n\end{aligned}$$

If the n^{th} product P_n is stable, we have $\lambda_n = 0$, and the equations give

$$\frac{d}{dt}(P_0 + P_1 + \dots P_n) = 0$$

hence the total mass remains constant, as it should do. It will be convenient, however, to consider the general case in which $\lambda_n \neq 0$.

To calculate the values of P_0 , P_1 , P_2 , ..., we observe that $e^{\lambda_0 t}$ is an integrating factor of the first equation, $e^{\lambda_1 t}$ an integrating factor of the second equation, and so on. Let us first assume as our supplementary conditions

$$P_0 = N_0, \quad P_1 = 0, \quad P_2 = 0, \dots P_n = 0 \quad \text{for } t = 0$$

The first equation gives $P_0 = C_0 e^{-\lambda_0 t}$, and since $P_0 = N_0$ for $t = 0$, we have $C_0 = N_0$, and so

$$P_0 = N_0 e^{-\lambda_0 t}$$

The second equation may be written in the form

$$\frac{d}{dt}(e^{\lambda_1 t} P_1) = \lambda_0 e^{\lambda_1 t} P_0 = \lambda_0 N_0 e^{(\lambda_1 - \lambda_0)t}$$

hence

$$e^{\lambda_1 t} P_1 = \frac{N_0 \lambda_0}{\lambda_1 - \lambda_0} [e^{(\lambda_1 - \lambda_0)t} + C_1]$$

Since $P_1 = 0$ when $t = 0$, we have $C_1 = -1$, and so

$$P_1 = \frac{N_0 \lambda_0}{\lambda_1 - \lambda_0} [e^{-\lambda_0 t} - e^{-\lambda_1 t}]$$

The second equation may now be written in the form

$$\frac{d}{dt}(e^{\lambda_2 t} P_2) = \lambda_1 e^{\lambda_2 t} P_1 = \frac{N_0 \lambda_0 \lambda_1}{\lambda_1 - \lambda_0} [e^{(\lambda_2 - \lambda_0)t} - e^{(\lambda_2 - \lambda_1)t}]$$

and gives

$$e^{\lambda_2 t} P_2 = \frac{\lambda_0 \lambda_1 N_0 e^{(\lambda_2 - \lambda_0)t}}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} + \frac{\lambda_0 \lambda_1 N_0 e^{(\lambda_2 - \lambda_1)t}}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} + C_2$$

* Cf. E. Rutherford, *Radio-activity*, 2nd edition, p. 330.

Since $P_2 = 0$ when $t = 0$, we find that

$$C_2 = \frac{N_0 \lambda_0 \lambda_1}{(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)}$$

$$\text{Hence } P_2 = N_0 \lambda_0 \lambda_1 \left[\frac{e^{-\lambda_0 t}}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} + \frac{e^{-\lambda_1 t}}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} \right]$$

We shall now prove by induction that

$$P_m = (-1)^m N_0 \lambda_0 \lambda_1 \dots \lambda_{m-1} \left[\frac{e^{-\lambda_0 t}}{f'_m(\lambda_0)} + \frac{e^{-\lambda_1 t}}{f'_m(\lambda_1)} + \dots \frac{e^{-\lambda_m t}}{f'_m(\lambda_m)} \right]$$

where $f_m(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1) \dots (\lambda - \lambda_m)$

and $f'_m(\lambda)$ denotes the derivative of $f_m(\lambda)$ with regard to λ .

Assuming that the law holds for a particular value of m , we have

$$\frac{d}{dt}(e^{\lambda_{m+1} t} P_{m+1}) = \lambda_m P_m e^{\lambda_{m+1} t} = (-1)^m N_0 \lambda_0 \lambda_1 \dots \lambda_m \sum_{s=0}^m \frac{e^{(\lambda_{m+1} - \lambda_s) t}}{f'_m(\lambda_s)}$$

Now, since

$$f'_m(\lambda_s) = (\lambda_s - \lambda_0)(\lambda_s - \lambda_1) \dots (\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1}) \dots (\lambda_s - \lambda_m)$$

we have $(\lambda_s - \lambda_{m+1}) f'_m(\lambda_s) = f'_{m+1}(\lambda_s)$

where $f_{m+1}(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1) \dots (\lambda - \lambda_{m+1})$

Hence

$$e^{\lambda_{m+1} t} P_{m+1} = (-1)^{m+1} N_0 \lambda_0 \lambda_1 \dots \lambda_m \left[\sum_{s=0}^m \frac{e^{(\lambda_{m+1} - \lambda_s) t}}{f'_{m+1}(\lambda_s)} + C_m \right]$$

Putting $t = 0$, we get

$$\sum_{s=0}^m \frac{1}{f'_{m+1}(\lambda_s)} + C_m = 0$$

but from a well-known theorem in the theory of equations

$$\sum_{s=0}^{m+1} \frac{1}{f'_{m+1}(\lambda_s)} = 0$$

Hence

$$C_m = \frac{1}{f'_{m+1}(\lambda_{m+1})}$$

and our formula now gives

$$P_{m+1} = (-1)^{m+1} N_0 \lambda_0 \lambda_1 \dots \lambda_m \sum_{s=0}^{m+1} \frac{e^{-\lambda_s t}}{f'_{m+1}(\lambda_s)}$$

Thus we have proved that if our formula holds for a given value of m , it holds for the next greater value of m ; but we have already shown that the formula holds for $m = 0, 1, 2$; hence it holds for all positive integral values of m .

The solution of the general case when the supplementary conditions are

$$P_0 = N_0, P_1 = N_1, \dots P_n = N_n \text{ for } t = 0$$

is obtained by adding together the solutions for the separate cases

$$P_0 = N_0, \quad P_1 = 0, \quad P_2 = 0, \dots, P_n = 0$$

$$P_0 = 0, \quad P_1 = N_1, \quad P_2 = 0, \dots, P_n = 0$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$P_0 = 0, \quad P_1 = 0, \quad \dots \dots \dots \quad P_n = N_n$$

Now it should be observed that if P_0 is zero initially, it is always zero; if both P_0 and P_1 are zero initially, they are always zero, and so on. Writing

$$F_r(\lambda) = (\lambda - \lambda_r)(\lambda - \lambda_{r+1}) \dots (\lambda - \lambda_m)$$

we find that

$$P_m = (-1)^m \left[N_0 \lambda_0 \lambda_1 \dots \lambda_{m-1} \sum_{s=0}^m \frac{e^{-\lambda_s t}}{F_0'(\lambda_s)} \right. \\ \left. - N_1 \lambda_1 \lambda_2 \dots \lambda_{m-1} \sum_{s=1}^m \frac{e^{-\lambda_s t}}{F_1'(\lambda_s)} + \dots (-1)^m N_m e^{-\lambda_m t} \right]$$

The case in which there is initially *radio-active equilibrium* is of some interest. We then have the relations

$$\lambda_0 N_0 = \lambda_1 N_1 = \dots \lambda_n N_n = a, \text{ say}$$

To sum the series, we make use of the relations

$$\frac{\lambda_s(\lambda_1 \lambda_2 \dots \lambda_{m-1})}{F_0'(\lambda_s)} = \frac{\lambda_0 \lambda_1 \dots \lambda_{m-1}}{F_0'(\lambda_s)} + \frac{\lambda_1 \dots \lambda_{m-1}}{F_1'(\lambda_s)} \\ \frac{\lambda_s(\lambda_2 \dots \lambda_{m-1})}{F_1'(\lambda_s)} = \frac{\lambda_1 \dots \lambda_{m-1}}{F_1'(\lambda_s)} + \frac{\lambda_2 \dots \lambda_{m-1}}{F_2'(\lambda_s)} \\ \dots \dots \dots \frac{\lambda_s}{F_s'(\lambda_s)} = \frac{\lambda_s}{F_s'(\lambda_s)}$$

$$\text{Whence } \lambda_s \left[\frac{\lambda_1 \lambda_2 \dots \lambda_{m-1}}{F_0'(\lambda_s)} - \frac{\lambda_2 \dots \lambda_{m-1}}{F_1'(\lambda_s)} + \dots \right] = \frac{\lambda_0 \lambda_1 \dots \lambda_{m-1}}{F_0'(\lambda_s)}$$

consequently it follows that the coefficient of $e^{-\lambda_s t}$ in the expression for P_m is

$$a \frac{\lambda_0 \lambda_1 \dots \lambda_{m-1}}{\lambda_s F_0'(\lambda_s)}$$

Hence, finally,

$$P_m = (-1)^m a \lambda_0 \lambda_1 \dots \lambda_{m-1} \sum_{s=0}^m \frac{e^{-\lambda_s t}}{\lambda_s F_0'(\lambda_s)}$$

EXAMPLES.

1. Prove that in the case when there is originally only one substance present,

$$\lambda_m \int_0^\infty P_m dt = N_0$$

and in the case when $P_0 = N_0, P_1 = N_1, \dots, P_m = N_m$ for $t = 0$,

$$\lambda_m \int_0^\infty P_m dt = N_0 + N_1 + \dots + N_m$$

What is the physical interpretation of this equation?

2. If there is initially only one substance present and

$$Q_m = \int_0^\infty e^{-xt} P_m dt \quad (x > 0)$$

prove that

$$Q_m = \frac{\lambda_0 \lambda_1 \dots \lambda_{m-1}}{(x + \lambda_0)(x + \lambda_1) \dots (x + \lambda_m)}$$

Hence show that the formula for P_m may be obtained by transforming this expression for Q_m into partial fractions.

§ 15. *A system of linear equations occurring in the theory of probability.* If in the last section $\lambda_0 = \lambda_1 = \dots \lambda_n$, the solution takes a different form. If the initial conditions are

$$P_0 = N_0, \quad P_1 = 0, \quad P_2 = 0, \dots P_n = 0$$

the value of P_0 is the same as before, but the equation to determine P_1 is

$$\frac{d}{dt} (e^{\lambda_0 t} P_1) = \lambda_0 e^{\lambda_0 t} P_0 = \lambda_0 N$$

$$\therefore e^{\lambda_0 t} P_1 = \lambda_0 N_0 t$$

Again,

$$\frac{d}{dt} (e^{\lambda_0 t} P_2) = \lambda_0 e^{\lambda_0 t} P_1 = \lambda_0^2 N_0 t$$

$$\therefore e^{\lambda_0 t} P_2 = \frac{(\lambda_0 t)^2}{2!} N_0$$

Proceeding in this way, we find that

$$P_m = N_0 e^{-\lambda_0 t} \frac{(\lambda_0 t)^m}{m!} \dots \dots \dots (1)$$

The solution for the case in which $P_0 = N_0, P_1 = N_1, \dots P_n = N_n$ may be found by superposition as before, and we have

$$P_m = e^{-\lambda_0 t} \left[N_0 \frac{(\lambda_0 t)^m}{m!} + N_1 \frac{(\lambda_0 t)^{m-1}}{(m-1)!} + \dots N_m \right]$$

A system of linear equations, in which the quantities λ are all equal, occurs in the solution of the following problem in the theory of probability.

Consider a large number of intervals and a large number of objects which are distributed at random among the intervals. Let $P_n(x)$ be the chance that an interval chosen at random contains exactly n objects when the average number of objects in an interval is x . To find $P_n(x)$, we shall suppose that the number of objects is increased by one, and that the number of intervals is a very large number x . The chance that the new object goes into the chosen interval is $\frac{1}{x}$, the chance that it does not is $\left(1 - \frac{1}{x}\right)$. The chance

$P_n\left(x + \frac{1}{x}\right)$ of having n objects in an interval chosen at random when the average number has increased to $x + \frac{1}{x}$ owing to the addition of the new object is the sum of two chances :

1°. The chance that the interval originally contained $n - 1$ objects and that the new object is included in the interval. This chance is

$$\frac{1}{x} P_{n-1}(x)$$

2°. The chance that the interval originally contained n objects and that the new object did not enter this particular interval. This chance is

$$\left(1 - \frac{1}{x}\right) P_n(x)$$

Hence we have the equation

$$P_n\left(x + \frac{1}{x}\right) = \left(1 - \frac{1}{x}\right) P_n(x) + \frac{1}{x} P_{n-1}(x)$$

Since $\frac{1}{x}$ is very small, we may write

$$P_n\left(x + \frac{1}{x}\right) - P_n(x) = \frac{1}{x} \frac{dP_n}{dx} + \text{higher powers of } \frac{1}{x}$$

hence we have approximately

$$\frac{dP_n}{dx} = -P_n + P_{n-1}$$

with

$$\frac{dP_0}{dx} = -P_0$$

When $x = 0$, we have $P_0 = 1$, $P_1 = P_2 = \dots P_n = 0$; accordingly our result (1) gives Poisson's formula

$$P_n = \frac{x^n}{n!} e^{-x}$$

EXAMPLES.

1. Prove that the probable value of n is x .
2. Prove that the probable value of the square of the deviation $(n - x)^2$ is x .

§ 16. *General theory of a system of linear equations.* Let us consider the system of linear equations :

$$\frac{dx_1}{dt} + a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = f_1(t)$$

$$\frac{dx_2}{dt} + a_{21}x_1 + \dots a_{2n}x_n = f_2(t)$$

$$\dots \dots \dots$$

$$\frac{dx_n}{dt} + a_{n1}x_1 + \dots a_{nn}x_n = f_n(t)$$

wherein each of the quantities a_{rs} is a function of t .

$$\begin{array}{rcl} (a_{11}-D)y_1 + a_{21}y_2 + \dots a_{n1}y_n & = & 0 \\ a_{12}y_1 + (a_{22}-D)q_2 + \dots a_{n2}y_n & = & 0 \\ . & . & . \\ a_{1n}y_1 + a_{2n}y_2 + \dots (a_{nn}-D)y_n & = & 0 \end{array}$$

The equation may then be integrated as it stands, and the solution is

$$u(x, y) = c$$

where c is an arbitrary constant. For instance, if M is a function of x only and N a function of y only, the equation is exact, and the solution is

$$\int M dx + \int N dy = c$$

It is clear that an equation of type (2) is not generally exact, for the relations

$$\frac{\partial u}{\partial x} = M \quad \frac{\partial u}{\partial y} = N \quad \dots \dots \dots (3)$$

imply that M and N are connected by the relation

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial M}{\partial y}$$

provided M , N , $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are continuous for some domain of the variables x and y .

Conversely, if the conditions just mentioned are satisfied and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, we can show that a function u exists for which equations (3) are satisfied.

Let us first of all integrate the equation $\frac{\partial u}{\partial x} = M$; regarding y as a constant, we get

$$u = \int M(a, y) da + Y$$

where Y , the constant of integration, is independent of x , but may be considered as a function of y . We shall endeavour to choose this function of y in such a way that the equation $\frac{\partial u}{\partial y} = N$ is satisfied.

Differentiating the previous equation with regard to y , we see that Y will satisfy the requirements if

$$N = \frac{\partial}{\partial y} \int M(a, y) da + \frac{dY}{dy}$$

Now, since $\frac{\partial M}{\partial y}$ is a continuous function of its arguments,* we may differentiate the integral under the integral sign by the rule of Leibnitz. This gives

$$\begin{aligned} N &= \int \frac{\partial}{\partial y} M(a, y) da + \frac{dY}{dy} \\ &= \int \frac{\partial}{\partial a} N(a, y) da + \frac{dY}{dy} \end{aligned}$$

since by hypothesis $\frac{\partial}{\partial y} M(a, y) = \frac{\partial}{\partial a} N(a, y)$

* See Goursat-Hedrick, *Mathematical Analysis*, Vol. I. p. 193.

Now $\frac{\partial}{\partial a} N(a, y)$ is a continuous function of a ; hence the integral may be evaluated, and we have

$$N(x, y) = N(x, y) - N(x_0, y) + \frac{dY}{dy}$$

where x_0 is the lower limit of the integral and is independent of y .

The function Y may now be obtained by solving the equation

$$\frac{dY}{dy} = N(x_0, y)$$

which is certainly soluble if x_0 belongs to the domain under consideration, for the function N is then a continuous function of y , and may consequently be integrated with regard to y .

When the equation (2) is not exact, it seems natural to try and make it exact by using a multiplier $\mu(x, y)$. This idea occurred to Clairaut (1739), and was developed by Euler, who obtained the partial differential equation

$$\frac{\partial}{\partial x} (\mu N) = \frac{\partial}{\partial y} (\mu M) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for the determination of μ . This equation is, of course, the condition that $\mu(M dx + N dy)$ should be exact.

If μ is one integrating factor and $u(x, y) = c$ a solution of the differential equation (2), it is easy to see that $\mu f(u)$ is also an integrating factor where f is an arbitrary function, for the equation

$$\frac{\partial}{\partial x} (\mu N f) = \frac{\partial}{\partial y} (\mu M f)$$

is a consequence of (3) and (4). By using different integrating factors, the solution of equation (2) is obtained in different equivalent forms. To understand the nature of this equivalence, we remark in the first place that if $u(x, y) = c$ is a solution of (2), then

$$F(u) = F(c)$$

is also a solution where F is an arbitrary function. Let us now see if there is a more general solution of the form

$$F(u, x) = a$$

where a is a constant. Differentiating with regard to x , we obtain

$$\frac{\partial F}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} \right) + \frac{\partial F}{\partial x} = 0$$

The quantity within brackets vanishes, since u satisfies equation (2); consequently $\frac{\partial F}{\partial x} = 0$; in other words, F depends only on u .

Sometimes the fact that two solutions of a differential equation

are equivalent leads to interesting analytical results. As an illustration, let us consider the exact equation

$$y dx + x dy = 0$$

which possesses the solution $u \equiv xy = c$. Since $\frac{1}{u} = \frac{1}{xy}$ is an integrating factor, we obtain the equation

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

whose solution

$$\log x + \log y = a$$

must be equivalent to $xy = c$. We may therefore write

$$\log x + \log y = F(xy)$$

Putting $y = 1$, we get $F(x) = \log x$, and consequently

$$\log x + \log y = \log (xy)$$

It should be remarked that if two relations

$$u(x, y) = a \quad v(x, y) = b$$

lead to the same differential equation of type (2), the relations

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

must be equivalent, and so

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 0$$

Conversely, if this relation holds, the two primitives lead to the same differential equation, and it follows from the above reasoning that v is a function of u .

It is generally impracticable to solve the partial differential equation for the determination of an integrating factor, but Euler has shown that the equation may be used with advantage to find when there is a multiplier of a specified type.

For instance, μ can be a function of some assigned quantity z depending on x and y if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N \frac{\partial z}{\partial x} - M \frac{\partial z}{\partial y}}$$

is a function of z . This is seen at once by putting $\mu = \varphi(z)$ in the partial differential equation (4). The cases in which $z = x, y, xy, \frac{y}{x}$ respectively are of chief interest. For further details the reader is referred to Boole's *Differential Equations*, Chap. IV.

Euler's partial differential equation is of some importance in thermodynamics. It is known that in certain heat exchanges

between two substances, the heat energy Q , which produces a small change in the state of one substance, can be represented by a simple formula of type

$$Q = l_v dv + \gamma_v d\theta$$

where θ is the absolute temperature and v some other variable, such as the volume of the substance. It is known, moreover, that θ^{-1} is a multiplier or integrating factor of the expression for Q . Euler's partial differential equation consequently leads to an important relation between the derivatives of the quantities l_v and γ_v . Another relation is obtained by using the first law of thermodynamics, which states that if p is the pressure of the substance $Q - p dv$ is the increase of the internal energy, and so is an exact differential.

EXAMPLES.

1. Prove that the following equations are exact :

$$(x^2 + y^2 - 1) dx + 2xy dy = 0$$

$$(x^2 - 3y) dx + (y^2 - 3x) dy = 0$$

2. Show that the relation

$$x + y = c(1 - xy)$$

leads to the differential equation

$$(1 + x^2) dy + (1 + y^2) dx = 0$$

Hence obtain the addition theorem for the function $\tan x$.

3. Show that the relation

$$x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$$

leads to the differential equation

$$\sqrt{1 - x^2} dy + \sqrt{1 - y^2} dx = 0$$

Hence obtain the addition theorem for the function $\sin x$.

4. Solve the equations

$$x^2 + y^2 + a \left(x \frac{dy}{dx} - y \right) = 0$$

$$(x^2 + y^2 + 2ax) dx + 2ay dy$$

where a is a constant.

5. The equation $M dx + N dy = 0$ possesses an integrating factor of the form $\mu = x^n F\left(\frac{y}{x}\right)$ if

$$\frac{x^2 \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) - nNx}{Mx + Ny}$$

is a function of $\frac{y}{x}$. Denoting this function by $\varphi\left(\frac{y}{x}\right)$, show that the function F is obtained by solving the equation

$$\frac{dF}{dz} + \varphi(z)F = 0$$

where $z = \frac{y}{x}$.

6. If $P(x) = y$ is a particular solution of the equation

$$\frac{dy}{dx} + y^2 = Q(x)$$

$\frac{1}{(y - P)^2} e^{-2 \int P dx}$ is an integrating factor (Lacroix).

7. It is known that the equation

$$(x \cos y - y \sin y) dy + (x \sin y + y \cos y) dx = 0$$

possesses an integrating factor which is a function of x only; find this integrating factor, and hence solve the equation.

8. Solve the equations

$$\begin{aligned} [y(x^2 + y^2) - xy] dx + x^2 dy &= 0 \\ (x^2 + y^2 + 1) dy + xy dx &= 0 \end{aligned}$$

§ 18. *Integrating factors of linear equations of the second order.* A linear differential equation of the second order may be written in the form

$$a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = f(x) \quad (1)$$

or in the abbreviated form $L_x(u) = 0$

where L_x denotes the operator by means of which the expression on the left-hand side of equation (1) is derived from u .

A quantity v is called an integrating factor of the expression $L_x(u)$ if we have identically

$$vL_x(u) \equiv \frac{d}{dx} \left[va \frac{du}{dx} + pu \right]$$

where p is some function which is independent of u . Comparing the two sides of this equation, we find that we must have

$$\frac{d}{dx}(va) + p = bv \quad \frac{dp}{dx} = cv$$

Eliminating the unknown function p , we find that v must satisfy the equation

$$\bar{L}_x(v) \equiv \frac{d^2}{dx^2}(av) - \frac{d}{dx}(bv) + cv = 0 \quad (2)$$

which is said to be *adjoint* to $L_x(u) = 0$.

When u and v are arbitrary functions of x , we have the identity

$$vL_x(u) - u\bar{L}_x(v) \equiv \frac{d}{dx} R(u, v) \quad (3)$$

where $R(u, v) = a \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) + uv \left(b - \frac{da}{dx} \right)$

It is clear from this relation that the expression $L_x(u)$ bears the same relation to $\bar{L}_x(v)$ as $\bar{L}_x(v)$ does to $L_x(u)$. The quantity R is called the *bilinear concomitant*, and $L_x(u)$, $\bar{L}_x(v)$ are said to be *adjoint differential expressions*. It is clear that a function u which satisfies the equation $L_x(u) = 0$ is an integrating factor of $\bar{L}_x(v)$.

When an integrating factor or solution of equation (2) is known, the equation (1) may be integrated. If we multiply by this factor v , a first integration gives

$$R(u, v) = \int v f dx + A$$

where A is an arbitrary constant. The last equation may be regarded as a linear equation of the first order for the determination of u . Multiplying throughout by the integrating factor

$$w = \frac{1}{a^2 v^2} e^{\int_a^b dx}$$

and integrating, we get

$$avwu = \int w dx \int v f dx + A \int w dx + B$$

where B is an arbitrary constant. The solution is thus of the form

$$u = Au_1(x) + Bu_2(x) + \frac{1}{avw} \int w dx \int v(\xi) f(\xi) d\xi$$

where $u_1(x)$ and $u_2(x)$ are particular solutions of the equation $L_x(u) = 0$.

The general solution of this last equation is, in fact, obtained by putting $f = 0$; it is $u = Au_1(x) + Bu_2(x)$

It is customary to say that the solution of equation (1) is made up of two parts, viz. the *complementary function* $Au_1(x) + Bu_2(x)$ and a *particular integral* involving the function f . It should be noticed that the general solution of equation (1) involves *two* arbitrary constants.

The two expressions $L(u)$, $\bar{L}(v)$ are identical if

$$\frac{d^2 a}{dx^2} - \frac{db}{dx} = 0 \quad 2 \frac{da}{dx} - b = b$$

i.e. if $\frac{da}{dx} = b$. In this case the equation $L(u) = 0$ is said to be *self-adjoint*. A self-adjoint equation may be written in the form

$$L_x(u) \equiv \frac{d}{dx} \left(a \frac{du}{dx} \right) + cu = 0$$

and relation (3) then becomes

$$vL_x(u) - uL_x(v) \equiv \frac{d}{dx} \left[a \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]$$

The general expression $L_x(u)$ of the second order may be made self-adjoint by multiplying it by a suitable factor z . This must be chosen so that

$$\frac{d}{dx}(az) = bz$$

The relation (3) may be extended to the case of a linear differential expression of the n^{th} order, such as

$$L_x(u) = a \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + a_n u$$

Since

$$av \frac{d^n u}{dx^n} = \frac{d}{dx} \left[av \frac{d^{n-1} u}{dx^{n-1}} - \frac{d}{dx} (av) \frac{d^{n-2} u}{dx^{n-2}} + \frac{d^2}{dx^2} (av) \frac{d^{n-3} u}{dx^{n-3}} - \dots \right] + (-1)^n u \frac{d^n}{dx^n} (av)$$

it follows that

$$\bar{L}_x(v) = (-1)^n \frac{d^n}{dx^n} (av) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (a_1 v) + \dots + a_n v$$

A function v , which satisfies the equation $\bar{L}_x(v) = 0$, is an integrating factor of the equation

$$L_x(u) = f(x)$$

Multiplying this equation by v and integrating, we obtain

$$R(u, v) = \int v f dx + C_1$$

where C_1 is an arbitrary constant. The new equation may be regarded as a linear differential equation of the $(n-1)^{\text{th}}$ order for the determination of u , and can be treated in a similar way if a suitable integrating factor can be found.

When n operations of this type are performed, an expression involving n arbitrary constants is finally obtained for u . This expression can be regarded as the sum of two expressions, a particular integral depending on the value of the function f , and a complementary function involving two arbitrary constants and independent of the form of f . This complementary function is a solution of the equation $L_x(u) = 0$.

EXAMPLES.

1. If $u(x)$ is a solution of the equation $a \frac{d^2 u}{dx^2} + b \frac{du}{dx} + cu = 0$, then $v = \frac{u}{a} e^{\int \frac{b}{a} dx}$ is a solution of the adjoint equation.

$$2. \text{ If } L_x(u) \equiv \left[\frac{d}{dx} + p(x) \right] \left[\frac{d}{dx} + q(x) \right] u = 0$$

prove that the adjoint equation is

$$\bar{L}_x(v) \equiv \left[\frac{d}{dx} - q(x) \right] \left[\frac{d}{dx} - p(x) \right] v = 0$$

This means that if the operator L_x can be broken up into factors, the adjoint operator $\bar{L}$ consists of the product of the adjoint factors

in the reverse order. Extend this theorem to the case in which there are n factors.

3. If $L_x(u)$ and $\bar{L}_x(v)$ are two adjoint expressions, the expression

$$\bar{L}_x L_x(u)$$

is self-adjoint, and the expression

$$\bar{L}_x \frac{d}{dx} L_x(u)$$

is self-adjoint except for a difference in sign (Darboux).

4. Prove that an equation of type

$$\frac{d^n}{dx^n} \left[A_n \frac{d^n u}{dx^n} \right] + \frac{d^{n-1}}{dx^{n-1}} \left[A_{n-1} \frac{d^{n-1} u}{dx^{n-1}} \right] + \dots \frac{d}{dx} \left[A_1 \frac{du}{dx} \right] + A_0 u = 0$$

is self-adjoint (Bertrand and Hesse).

§ 19. *Equations of various types.* A differential equation of the form

$$f\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0$$

is said to be *exact* when we have identically

$$f\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) = \frac{d}{dx} g\left(\frac{dy}{dx}, y, x\right)$$

In this case the equation

$$g\left(\frac{dy}{dx}, y, x\right) = c$$

is a *first integral*; if this equation of the first order can be solved, the original equation can also be solved.

Sometimes an equation may be made exact by multiplying throughout by an integrating factor; for instance, the equation

$$\frac{d^2 y}{dx^2} + \mu^2 \sin y = 0 \quad \dots \quad (1)$$

may be made exact by multiplying throughout by $\frac{dy}{dx}$, the resulting equation gives on integration

$$\frac{1}{2} \left(\frac{dy}{dx}\right)^2 - \mu^2 \cos y = c$$

In this equation the variables are separated, and so the integration can be carried a step further.

The differential equation (1) is the equation of motion of a point mass which moves in a circle under gravity. In fact, if θ is the angle which the radius to the moving point makes with the vertical, the tangential acceleration is $a\ddot{\theta}$ where a is the radius of the circle. Equating this to the resolved part of the acceleration due to gravity, viz. $-g \sin \theta$, we obtain the equation

$$a\ddot{\theta} = -g \sin \theta$$

which is of the form (1) except that θ and t replace y and x .

The method of integrating factors is particularly useful in the case of the differential equations occurring in dynamics, and can be employed when a system of differential equations is to be solved.

Let us consider, for instance, the equations

$$\frac{d^2x}{dt^2} = \frac{\partial R}{\partial x} \quad \frac{d^2y}{dt^2} = \frac{\partial R}{\partial y} \quad \frac{d^2z}{dt^2} = \frac{\partial R}{\partial z} \quad \dots \quad (2)$$

where $R = f(x^2 + y^2 + z^2)$. These are the equations of motion of a particle under the influence of acting from a fixed centre and varying as some function of the distance.

Multiplying the equations by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ respectively and adding, we get

$$\frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dy}{dt} \frac{d^2y}{dt^2} + \frac{dz}{dt} \frac{d^2z}{dt^2} = \frac{\partial R}{\partial x} \frac{dx}{dt} + \frac{\partial R}{\partial y} \frac{dy}{dt} + \frac{\partial R}{\partial z} \frac{dz}{dt} = \frac{dR}{dt}$$

Hence a first integral is

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = R + C$$

This is the *energy-equation*. To obtain some further integrals we observe that

$$\frac{\partial R}{\partial y} = 2y f'(x^2 + y^2 + z^2) \quad \frac{\partial R}{\partial z} = 2z f'(x^2 + y^2 + z^2)$$

hence
$$y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2} = 0$$

This is an exact equation whose first integral is

$$\left. \begin{aligned} y \frac{dz}{dt} - z \frac{dy}{dt} &= \alpha \\ \text{Similarly, we may obtain two other} \\ \text{integrals} \quad z \frac{dx}{dt} - x \frac{dz}{dt} &= \beta \\ x \frac{dy}{dt} - y \frac{dx}{dt} &= \gamma \end{aligned} \right\} \dots \dots \dots (3)$$

where α, β, γ are arbitrary constants. These equations imply that the particle's moments of momentum round the different axes remain constant.

Squaring the last three equations and adding, we get

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= (x^2 + y^2 + z^2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (x\dot{x} + y\dot{y} + z\dot{z})^2 \\ &= 2r^2(R + c) - (r\dot{r})^2 \quad \text{where } r^2 = x^2 + y^2 + z^2 \end{aligned}$$

Writing $\alpha^2 + \beta^2 + \gamma^2 = a^2$, we have

$$\begin{aligned} r \frac{dr}{dt} &= [2r^2(R + c) - a^2]^{\frac{1}{2}} \\ \therefore t - t_0 &= \int \frac{r dr}{[2r^2(R + c) - a^2]^{\frac{1}{2}}} \end{aligned}$$

It follows from equations (3) that $\alpha x + \beta y + \gamma z = 0$, consequently the point x, y, z always lies in a fixed plane through the origin. Let us transform the axes so that this plane is $z = 0$. The equations (1) then have the same general form as before, but we have the additional information that $z^* = 0$. The only equation of the set (3) which remains is now

$$x^* \frac{dy^*}{dt} - y^* \frac{dx^*}{dt} = \gamma^* = a$$

since α^* and β^* are zero in the new system of coordinates.

Dividing out by r^2 , which is now equal to $x^{*2} + y^{*2}$, since z^* is zero, we get

$$\frac{x^* \frac{dy^*}{dt} - y^* \frac{dx^*}{dt}}{x^{*2} + y^{*2}} = \frac{a}{r^2}$$

$$\therefore \tan^{-1} \frac{y^*}{x^*} = \int \frac{a dt}{r^2} = \int \frac{a dr}{r[2r^2(R+c) - a^2]} + b$$

where b is an arbitrary constant.

Writing $y^* = x^* \tan \varphi$ where φ denotes the quantity on the right-hand side of the last equation, we have $x^* = r \cos \varphi$, $y^* = r \sin \varphi$. To obtain the complete solution of the original set of differential equations we must transform back to the original set of axes. The arbitrary constants appearing in the solution are t_0, a, b, c and the ratios of α, β, γ ; six arbitrary constants altogether. This number may be obtained by adding together the orders of the three equations ($2 + 2 + 2 = 6$).

EXAMPLES.

1. Solve the equation $\frac{d^2 u}{dt^2} + u = \frac{f(u)}{h^2 u^2}$

where h is a constant.

2. Solve the equations

$$(x^2 + y^2)^{\frac{3}{2}} \frac{d^2 x}{dt^2} + \mu x = 0$$

$$(x^2 + y^2)^{\frac{3}{2}} \frac{d^2 y}{dt^2} + \mu y = 0$$

3. If $p = \frac{dy}{dx}$, $r = \frac{d^2 y}{dx^2}$, the equation

$$rf(x, y, p) + g(x, y, p)$$

is exact if

$$\frac{\partial^2 g}{\partial p^2} = p \frac{\partial^2 f}{\partial p \partial y} + \frac{\partial^2 f}{\partial p \partial x} + 2 \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 g}{\partial p \partial x} + p \frac{\partial^2 g}{\partial p \partial y} - \frac{\partial g}{\partial y} = \frac{\partial^2 f}{\partial x^2} + 2p \frac{\partial^2 f}{\partial x \partial y} + p^2 \frac{\partial^2 f}{\partial y^2}$$

4. Solve the equations

$$2xr = p$$

$$r(2xp + y) + p^2 = p \frac{y}{x}$$

$$yr + p^2 = y^2 + 2xpy$$

$$y^2r + 1 = 0$$

$$r(y^2 - x^2) + (1 - p^2)(y - xp) = 0$$

$$r(1 - x) = p(1 + p)$$

$$ry = p^2$$

$$yr(y^2 - x^2) = p^2(x^2 + y^2) - 2xyp$$

$$yr = 1 + p^2$$

5. Solve the equations

$$\ddot{\xi} = -\lambda \dot{y} \cos \kappa \xi \quad \ddot{y} = \lambda \dot{\xi} \cos \kappa \xi$$

where λ and κ are constants. If $\xi = x + Vt$, these equations determine the motion of an electron under the influence of a plane wave of light moving with velocity V parallel to the axis of x . (J. H. Jeans.)

6. The motion of a certain dynamical system is determined by the differential equation

$$\frac{d^2x}{dt^2} = (\cos x - a) \sin x$$

where a denotes a positive constant. Find the positions of stable equilibrium and the duration of small oscillations round these positions for different values of a .

CHAPTER III

TRANSFORMATIONS

§ 20. *Preliminary remarks.* It is sometimes possible to transform a given differential equation into an equation which can be integrated directly. In many cases the new equation is derived from the original one by a change of variables, sometimes it is derived from it by differentiation. It is generally possible to use a solution of the new equation to obtain a solution of the original equation. If, however, the new equation cannot be solved in a simple manner by means of known functions, we must not conclude that the transformation is of no value, because a differential equation is frequently used to define a new function, and then it is important to know which types of differential equations can be solved with the aid of the new function.

Transformations which change a given differential equation into another differential equation of the same general type are of some interest, as they frequently indicate properties of the functions defined by the given differential equation.

§ 21. *Homogeneous equations of the first type.* The differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \dots \dots \dots (1)$$

is said to be homogeneous because it is unaltered when x, y are replaced by kx, ky respectively, k being an arbitrary constant. This means that the equation is covariant under the group of transformations $x' = kx, y' = ky$. The transformations of this type are said to form a *group*, because successive applications of two transformations of this type with arbitrary values k_1, k_2 of k are equivalent to a single transformation of this type. In other words, if $x' = k_1x, y' = k_1y$ and $x'' = k_2x', y'' = k_2y'$, then $x'' = k_1k_2x, y'' = k_1k_2y$.

To solve the equation we put $y = xt$, then $\frac{dy}{dx} = x \frac{dt}{dx} + t$, and the equation becomes

$$x \frac{dt}{dx} + t = f(t)$$

The variables are now separated, and the solution is

$$\int \frac{dx}{x} = \int \frac{dt}{f(t) - t} + c$$

The differential equation (1) frequently arises in geometrical problems in the following way :

Let us suppose that a curve is defined by some geometrical relation connecting the angle ψ , which the tangent makes with a fixed line, with the angle θ , which the radius vector from a fixed point O makes with this line.

If the relation in question is

$$\tan \psi = f(\tan \theta) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it is equivalent to the differential equation (1).

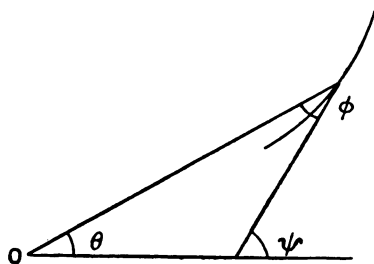


FIG. 4.

This equation may also be solved by transforming to polar co-ordinates r, θ , for we have

$$r \frac{d\theta}{dr} = \tan \varphi = \tan (\psi - \theta) = \frac{f(\tan \theta) - \tan \theta}{1 + \tan \theta f(\tan \theta)}$$

or

$$\frac{dr}{r} = \frac{1 + \tan \theta f(\tan \theta)}{f(\tan \theta) - \tan \theta} d\theta$$

The solution is evidently of the form

$$r = Ae^{G(\theta)}$$

where A is an arbitrary constant. The differential equation evidently corresponds to a class of curves which are similar and similarly situated, the point O being the centre of similitude.

Sometimes the relation between θ and ψ cannot be expressed very simply in the form (2), but may be written in the form

$$\tan \theta = F(\tan \psi)$$

In this case the differential equation is

$$y = x F\left(\frac{dy}{dx}\right)$$

To solve this equation we take $\frac{dy}{dx} = p$ as a new variable. Differentiating the equation $y = xF(p)$ with regard to x , we get

$$p = F(p) + xF'(p) \frac{dp}{dx}$$

whence $\frac{dx}{x} = \frac{F'(p) dp}{p - F(p)}$

The solution is now obtained by eliminating p from the equations

$$\log x = \int \frac{F'(p) dp}{p - F(p)} + C \quad y = xF(p)$$

wherein C is an arbitrary constant. We may, however, retain p and regard the last two equations as equations giving a parametric representation of the coordinates of points on the curve in terms of the parameter p .

If the relation between θ and ψ is of the form

$$\tan \theta = f[\tan \varphi] = f[\tan (\psi - \theta)]$$

and the resulting equation cannot be easily solved for $\frac{dy}{dx}$ or $\frac{y}{x}$, it is convenient to use polar coordinates. The differential equation is then

$$\tan \theta = f\left[r \frac{d\theta}{dr}\right] = f(\tan \varphi)$$

Differentiating with respect to r , we get

$$\sec^2 \theta \frac{d\theta}{dr} = f'(\tan \varphi) \sec^2 \varphi \frac{d\varphi}{dr}$$

or $(1 + f^2) \tan \varphi = r f'(\tan \varphi) \sec^2 \varphi \cdot \frac{d\varphi}{dr}$

The variables r , φ may now be separated, and we obtain

$$\int \frac{dr}{r} = \int \frac{f'(\tan \varphi) \sec^2 \varphi d\varphi}{\tan \varphi (1 + f^2)} + C$$

Denoting this relation by $r = AG(\varphi)$, we can express x and y in terms of φ .

EXAMPLES.

1. Solve the equations

$$(y^2 - xy) dx + x^2 dy = 0$$

$$y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$x + y \frac{dy}{dx} = 2y$$

$$y = xp^2$$

2. Find the general equations of curves that are specified by the following properties, a and b being constants:

(1) $\psi = 2\theta$.

(4) $\tan \psi = a + b \tan \theta$.

(2) $\psi = \pi - \theta$.

(5) $\tan \psi = a + b \tan \varphi$.

(3) $\varphi = a$.

(6) $\tan \theta = a + b \tan \varphi$.

3. Prove that the equation

$$\varphi \left(\frac{y}{x}\right) dx + \psi \left(\frac{y}{x}\right) dy + x^n \chi \left(\frac{y}{x}\right) (x dy - y dx) = 0$$

may be solved by making the substitution $y = vx$ and using the method of § 8 (G. Boole).

§ 22. *Equations which can be reduced to homogeneous equations.* If a and b are constants, the equation

$$\frac{dy}{dx} = F\left(\frac{y+b}{x+a}\right) \dots \dots \dots (1)$$

can evidently be reduced to the homogeneous form by putting

$$x' = x + a \qquad y' = y + b$$

This substitution is successful in the case of the equation

$$(lx + my + n)dx + (\lambda x + \mu y + \nu)dy = 0 \dots (2)$$

where $l, m, n, \lambda, \mu, \nu$ are constants. The transformed equation will be homogeneous if a and b are chosen, so that

$$la + mb = n \qquad \lambda a + \mu b = \nu$$

These equations fail to determine a and b when $\frac{l}{\lambda} = \frac{m}{\mu}$. In this case we write $\lambda = kl, \mu = km$, and take $lx + my$ as a new variable t . Since $l dx + m dy = dt$, the differential equation becomes

$$m(t + n)dx + (kt + \nu)(dt - l dx) = 0$$

or
$$\frac{dx}{dt} = \frac{kt + \nu}{l(kt + \nu) - m(t + n)}$$

The integration can now be effected.

EXAMPLES.

1. Prove that the differential equation (2) can also be solved by writing it in the form

$$\frac{dx - \theta_1 dy}{(\lambda + \theta_1 l)x + (\mu + \theta_1 m)y + \nu + \theta_1 n} = \frac{dx - \theta_2 dy}{(\lambda + \theta_2 l)x + (\mu + \theta_2 m)y + \nu + \theta_2 n}$$

and choosing θ_1, θ_2 so that they satisfy the equation

$$l\theta^2 + (\lambda + m)\theta + \mu = 0$$

When does this method break down?

2. Solve the equations

$$\begin{aligned} (x - y - 1)dx + (x - y + 1)dy &= 0 \\ (x - 3y + 2)dx + (3x - y + 6)dy &= 0 \end{aligned}$$

3. Solve the equation

$$\frac{dy}{dx} = 1 + \frac{m_1 m_2}{(m_1 - m_2)^2} \frac{x - a}{y}$$

wherein a, m_1, m_2 are constants.

§ 23. *Clairaut's equation.* The equation

$$y = px + f(p) \dots \dots \dots (1)$$

in which $p = \frac{dy}{dx}$ may be solved by differentiation. The resulting equation is

$$[x + f'(p)] \frac{dp}{dx} = 0$$

The transformed equation is thus $\frac{dp}{dx} = 0$, and this gives $p = c$ where c is a constant. Substituting in (1), we obtain the *general solution*

$$y = cx + f(c) \quad \dots \dots \dots (2)$$

A *singular solution* is obtained by putting $x + f'(p) = 0$, and eliminating p between this equation and (1). This solution does not involve an arbitrary constant, when interpreted geometrically it represents the envelope of the family of lines given by (2). The theory of singular solutions will be discussed more fully in Chapter IV.

Equation (1) is usually called Clairaut's equation if it is written in the more general form

$$F(y - px, p) = 0$$

the general solution is $F(y - cx, c) = 0$

§ 24. *Equations of the first order which can be solved for either x or y.* When an equation can be written in the form

$$y = F(x, p) \quad \dots \dots \dots (1)$$

it may sometimes be solved by differentiation. The resulting equation is

$$p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \frac{dp}{dx}$$

and can be regarded as a differential equation for p . If this new equation can be solved, giving

$$f(p, x, c) = 0 \quad \dots \dots \dots (2)$$

where c is a constant, the solution of the original equation may be obtained by eliminating p between (1) and (2).

When an equation can be written in the form

$$x = F(y, p) \quad \dots \dots \dots (3)$$

we must naturally differentiate with regard to y . The resulting equation is

$$\frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p} \frac{dp}{dy}$$

If this equation in p and y can be solved, giving

$$f(p, y, c) = 0 \quad \dots \dots \dots (4)$$

a solution of the original equation is obtained by eliminating p between (3) and (4).

EXAMPLES.

1. Solve the equations

$$y = p^2 f(x)$$

$$y = p^n f(x)$$

$$x = p^n f(y)$$

$$x + yp = ap^2$$

$$y = f(xp)$$

2. Solve the equation $y = xf(p) + \varphi(p)$ and consider the case when $f(p)$ and $\varphi(p)$ are polynomials with a common factor p^n .

§ 25. *Equations of the second order which do not contain x explicitly.* An equation of the second order which does not contain x explicitly may be written in the form

$$\psi\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To solve this equation we write

$$\frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} = p \frac{dp}{dy}$$

The equation then takes the form

$$\psi\left(y, p, p \frac{dp}{dy}\right) = 0$$

and can be regarded as a differential equation of the first order to determine p as a function of y . If a solution of this equation can be obtained in the form

$$p = f(y, a)$$

where a is an arbitrary constant, the original equation possesses the solution

$$x = \int \frac{dy}{f(y, a)} + c$$

which depends on two arbitrary constants, a and c .

Differential equations of type (1) are of frequent occurrence in dynamics when the acceleration of a particle depends on its position and velocity but does not involve the time explicitly. An equation of type (1) also arises when an attempt is made to derive the general equation of a curve from some property involving the radius of curvature. Let ξ, η be the perpendiculars from a point O on the tangent and normal at a point P to a curve, ψ the angle which the tangent makes with a fixed line, and ρ the radius of curvature at P . It is known that

$$\rho = \xi + \frac{d^2\xi}{d\psi^2} \qquad \eta = \frac{d\xi}{d\psi}$$

Hence, if we are given that $\rho = F(\xi, \eta)$, ξ satisfies the differential equation

$$\frac{d^2\xi}{d\psi^2} + \xi = F\left(\xi, \frac{d\xi}{d\psi}\right)$$

Let $\xi = f(\psi, a, b)$ be the solution of this equation, then ρ is determined as a function of ψ , and x, y may be expressed in terms of ψ with the aid of the equations

$$x = \int \rho \cos \psi \, d\psi \qquad y = \int \rho \sin \psi \, d\psi$$

EXAMPLES.

1. Solve the equations

$$\begin{aligned}\frac{d^2y}{dx^2} + \mu^2 y + \lambda \left(\frac{dy}{dx}\right)^2 &= 0 \\ \frac{d^2y}{dx^2} + 2\kappa \frac{dy}{dx} + \lambda \left(\frac{dy}{dx}\right)^2 &= 0 \\ \frac{d^2y}{dx^2} + y &= \lambda \sqrt{y^2 + \left(\frac{dy}{dx}\right)^2}\end{aligned}$$

where λ , μ , κ are arbitrary constants.

§ 26. *Equations of the second order which do not involve y explicitly.*
The equation

$$\varphi\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

may be replaced by an equation of the first order by writing $\frac{dy}{dx} = p$.

If the equation

$$\varphi\left(x, p, \frac{dp}{dx}\right) = 0$$

can be solved and p expressed in the form

$$p = f(x, a)$$

where a is a constant, a solution of the original equation is given by

$$y = \int f(x, a) dx + c$$

and involves two arbitrary constants a and c .

An equation of the present type can evidently be reduced to an equation of the preceding type by taking x as the dependent variable.

EXAMPLES.

1. Solve the equation
- $\frac{dy}{dx} = x \frac{d^2y}{dx^2} - \left(\frac{d^2y}{dx^2}\right)^2$

and show that in addition to the general solution involving two arbitrary constants there is a singular solution involving one arbitrary constant.

2. Solve the equations
- $(x + p) \frac{dp}{dx} = p$

$$x^2 \frac{dp}{dx} + p^2 - xp = 0$$

§ 27. *Equations of the second order which may be solved by differentiation.* It is hardly possible to write down the general form of a differential equation of the second order which can be solved by differentiation. A simple example will be sufficient to illustrate the method.

Consider the equation

$$2y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = f \left[\frac{dy}{dx} - x \frac{d^2y}{dx^2} \right]$$

where f is an arbitrary function. Differentiating with regard to x , we obtain

$$2y \frac{d^3y}{dx^3} = -x \frac{d^3y}{dx^3} f' \left[\frac{dy}{dx} - x \frac{d^2y}{dx^2} \right]$$

hence either $\frac{d^3y}{dx^3} = 0$ or

$$2y + x f' \left[\frac{dy}{dx} - x \frac{d^2y}{dx^2} \right] = 0$$

In the first case we have $y = ax^2 + bx + c$, and, on substituting in the differential equation, we find that the constants a, b, c must be connected by the relation

$$4ac - b^2 = f(b)$$

In the second case we may write

$$\frac{dy}{dx} - x \frac{d^2y}{dx^2} = F \left(\frac{2y}{x} \right) \quad 2y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx} \right)^2 = G \left(\frac{2y}{x} \right)$$

where F and G are certain functions depending on f .

Eliminating $\frac{d^2y}{dx^2}$, we obtain a relation of the form

$$\varphi \left(\frac{dy}{dx}, \frac{y}{x} \right) = 0$$

This is a homogeneous equation and may be solved by the method of § 21.

EXAMPLES.

1. Solve the equation

$$(a + bx + 2cy) \frac{d^2y}{dx^2} = b \frac{dy}{dx} + c \left(\frac{dy}{dx} \right)^2$$

where a, b, c are constants.

2. Solve the equation

$$\frac{1}{3} \left(\frac{dy}{dx} \right)^3 - y \frac{dy}{dx} \frac{d^2y}{dx^2} + y f'(y) - f(y) = F \left[\frac{dy}{dx} \frac{d^2y}{dx^2} - f'(y) \right]$$

where $f(y), F(y)$ are arbitrary functions.

§ 28. *Equations of Lagrange's type.* The equation considered in the last paragraph is of the form

$$u = f(v) \quad \dots \dots \dots (1)$$

where the equations $\frac{du}{dx} = 0, \frac{dv}{dx} = 0$ contain a common factor and are satisfied in virtue of a single differential equation, which, of course, does not depend upon the form of f . Equations which can be thrown into such a form are said to be of Lagrange's type.

It is instructive to ascertain how an equation of Lagrange's type arises from a given primitive. For simplicity we shall consider the case of an equation of the first order.

Let the given primitive be

$$\varphi(x, y, a, b) = 0 \quad (2)$$

where the constants a, b are connected by a relation

$$a = f(b)$$

We shall commence by discarding this relation and treating the constants a and b in (2) as independent. Under this understanding we may eliminate a and b in turn from the two equations

$$\varphi = 0 \quad \frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0$$

Let us suppose that the resulting equations can be written in the form

$$u(x, y, p) = a \quad \text{and} \quad v(x, y, p) = b \quad (3)$$

respectively. When we differentiate these equations, two differential equations of the second order are obtained. If these are equivalent, equation (1) is of Lagrange's type, and its solution is obtained by eliminating p from the equations (3).

The two differential equations of the second order can generally be expected to be equivalent, since they both have (2) as a primitive.

EXAMPLES.

Solve the equations $y^2(1 + p^2) = f(x + yp)$

$$yp = xf(y^2 - xyp)$$

$$yp - x = f(y^2 - y^2p^2)$$

$$y - 2xp = f(xp^3)$$

$$F[y - G(p) + pG'(p), x + G'(p)] = 0$$

$$\frac{y - xp}{y^2 + p} = f\left(\frac{y - xp}{1 + x^2p}\right)$$

where $p = \frac{dy}{dx}$ and f, F, G are arbitrary functions.

§ 29. *Riccati's equation.* An equation of type

$$\frac{dy}{dx} = Py^2 + Qy + R \quad (1)$$

where P, Q, R are functions of x is now generally known as Riccati's equation after Count Riccati, who studied the particular equation

$$\frac{dy}{dx} + by^2 = cx^m \quad (2)$$

where b and c are constants.

If one solution $y = u(x)$ of equation (1) is known, the equation may be integrated by writing $y = u + \frac{1}{v}$ and regarding v as a new dependent variable. The transformed equation is

$$\frac{dv}{dx} + 2(Pu + Q)v + P = 0$$

and may be solved by the method of § 7. Since the equation is linear, the general solution is of the form

$$v = cf(x) + g(x)$$

where c is an arbitrary constant and $f(x)$, $g(x)$ are independent of c .

The complete solution of Riccati's equation is now seen to be of the form

$$y = u + \frac{1}{cf(x) + g(x)} = \frac{cF(x) + G(x)}{cf(x) + g(x)} \quad (3)$$

If F , G , f , g are single-valued functions of x , y is also a single-valued function of x , and if we are given that $y = y_0$ when $x = x_0$, there is only one value of c , and so y is determined uniquely.

If the functions F , G , f , g are not all single-valued functions of x , for instance, if $f(x) = \sqrt{x^2 - 1}$, y is generally a multiform function of x , but its branch points, i.e. the points at which two values of the function become the same, depend only on the positions of the branch points of the functions F , G , f , g and not on the arbitrary constant c , and are consequently the same for all solutions of the equation. In the present case $x = 1$ and $x = -1$ are branch points of $f(x)$, and are consequently branch points of y unless the radical $\sqrt{x^2 - 1}$ disappears when the expression (3) is simplified. At any rate, we may conclude that a solution of Riccati's equation has *fixed branch points*.

This is not true for every differential equation of the first order, for instance, the general solution of the equation

$$2y \frac{dy}{dx} = 1$$

is $y = \sqrt{x - c}$. This is a two-valued function with a branch point at $x = c$. The position of the branch point thus varies with c , and is consequently different for different solutions of the equation. In other words, $x = c$ is a *movable branch point*.

Euler has shown that Riccati's equation may be transformed into a linear differential equation of the second order by making the substitution

$$y = -\frac{1}{Pz} \frac{dz}{dx}$$

The resulting equation is

$$P \frac{d^2z}{dx^2} - \left(PQ + \frac{dP}{dx} \right) \frac{dz}{dx} + P^2 R z = 0$$

In particular, equation (2) can be replaced by the equation

$$\frac{d^2z}{dx^2} + bcx^mz = 0$$

An important property of Riccati's equation is that the cross ratio of four particular solutions is independent of x . We easily find from (3) that if y_1, y_2, y_3, y_4 are the solutions corresponding to the constants c_1, c_2, c_3, c_4 ,

$$\frac{(y_2 - y_3)(y_1 - y_4)}{(y_3 - y_1)(y_2 - y_4)} = \frac{(c_2 - c_3)(c_1 - c_4)}{(c_3 - c_1)(c_2 - c_4)}$$

This result is due to E. Weyr and E. Picard.

EXAMPLES.

1. A gas is being ionised by an ionising agent, and at time t there are n_1 free positive ions and n_2 free negative ions per cubic centimetre. If q ions per cubic centimetre are produced every second by the agent, and the number of recombinations per second is proportional to n_1n_2 , we have the equations

$$\frac{dn_1}{dt} = q - \alpha n_1 n_2$$

$$\frac{dn_2}{dt} = q - \alpha n_1 n_2$$

Solve these equations on the supposition that α and q are constants.

2. When can the equation

$$\frac{d^2y}{dx^2} = Py^3 + 3Qy^2 + 3Ry + S$$

be satisfied by all the solutions of an equation of type

$$\frac{dy}{dx} = ay^2 + by + c?$$

Obtain the necessary and sufficient relation between the functions P, Q, R, S .

3. The equation

$$\frac{dy}{dx} + f(x) \sin y + g(x) \cos y + h(x) = 0$$

can be reduced to a Riccatian equation by means of the substitution $z = \tan \frac{y}{2}$.

4. Integrate the equations

$$\frac{dy}{dx} + \frac{y}{x} + y^2 = 0$$

$$\frac{dy}{dx} - \frac{y}{x} + y^2 = 0$$

5. Integrate the equation

$$x^3 \frac{dy}{dx} - y^3 - x^3 y + x^3 = 0$$

by making use of the fact that $y = x$ is a particular solution.

6. Prove that equation (1) is transformed into another Riccatian equation by the substitution

$$y = \frac{AY + B}{CY + D}$$

where A, B, C, D are functions of x .

7. Prove that if A, B, C, p , q , r are arbitrary functions of t and $l^2 + m^2 + n^2 = 1$, the equations

$$\frac{dl}{dt} = l(Al + Bm + Cn) + qn - rm - A$$

$$\frac{dm}{dt} = m(Al + Bm + Cn) + rl - pn - B$$

$$\frac{dn}{dt} = n(Al + Bm + Cn) + pm - ql - C$$

can be reduced to two Riccatian equations in s and σ by putting $s(1 + n) = l + im$, $\sigma(1 + n) = l - im$.

§ 30. *The linear differential equation of the second order.* If the differential equation

$$L_x(u) \equiv a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = f(x) \quad (1)$$

is transformed by the substitution $u = zw$ and z taken as new dependent variable, the resulting equation is

$$aw \frac{d^2 z}{dx^2} + \left(2a \frac{dw}{dx} + bw\right) \frac{dz}{dx} + \left(a \frac{d^2 w}{dx^2} + b \frac{dw}{dx} + cw\right) z = f(x)$$

The function w is at our disposal if we choose it so that $L_x(w) = 0$; the new equation may be written in the form

$$\frac{d^2 z}{dx^2} + \left(\frac{2}{w} \frac{dw}{dx} + \frac{b}{a}\right) \frac{dz}{dx} = \frac{f}{aw}$$

This is a linear equation of the first order for $\frac{dz}{dx}$, and an integrating factor is consequently $w^2 \exp\left(\int \frac{b}{a} dx\right)$. Integrating, we obtain

$$w^2 \frac{dz}{dx} e^{\int \frac{b}{a} dx} = \int \frac{f}{w} e^{\int \frac{b}{a} dx} + c$$

The value of z may now be obtained by integration; consequently, if one solution of the equation $L_x(w) = 0$ is known, equation (1) can be solved. When a solution of the equation $L_x(w) = 0$ cannot be readily obtained, it is sometimes useful to reduce (1) to a normal

form in which the second term is missing. This may be done by choosing w so that

$$2a \frac{dw}{dx} + bw = 0$$

Substituting this value of w , we obtain the equation

$$\frac{d^2z}{dx^2} + Iz = \frac{f}{aw}$$

where

$$I = \frac{c}{a} - \frac{b^2}{4a^2} - \frac{1}{2a} \frac{db}{dx} + \frac{b}{2a^2} \frac{da}{dx}$$

This quantity I is the same for all linear differential equations which can be derived from (1) by a substitution of the form $u = zv$. For, if

$$a' \frac{d^2z}{dx^2} + b' \frac{dz}{dx} + c'z = g$$

is such an equation, we must have

$$\lambda a' = av \quad \lambda b' = 2a \frac{dv}{dx} + bv \quad \lambda c' = a \frac{d^2v}{dx^2} + b \frac{dv}{dx} + cv$$

where λ is some function of x ;

$$\therefore I' = \frac{c'}{a'} - \frac{b'^2}{4a'^2} - \frac{1}{2a'} \frac{db'}{dx} + \frac{b'}{2a'^2} \frac{da'}{dx} = \frac{c}{a} - \frac{b^2}{4a^2} - \frac{1}{2a} \frac{db}{dx} + \frac{b}{2a^2} \frac{da}{dx} = I$$

I is on this account called the *invariant* for substitutions of type $u = vz$.

We may also get rid of the second term in an equation by making a change of independent variable.

Let us first of all reduce the equation to the self-adjoint form

$$\frac{d}{dx} \left(p \frac{du}{dx} \right) + qu = F(x)$$

by writing $va = p$, $vb = \frac{dp}{dx}$, $vc = q$, $vf = F$, where v is a solution of

$$\frac{d}{dx} (va) = vb$$

Now define the new variable t by the equation $t = \int \frac{dx}{p}$, and express pq , Fp in terms of t . The resulting equation is

$$\frac{d^2u}{dt^2} + pqu = Fp$$

and the coefficient of u is

$$pq = \frac{c}{a} e^{2 \int \frac{b}{a} dx}$$

EXAMPLES.

1. Reduce the equation

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + n^2 u = 0$$

to the canonical form $\frac{d^2 u}{dt^2} + n^2 u = 0$

and hence solve the equation when n is a constant

2. Reduce the equation

$$\frac{d^2 u}{dx^2} + \frac{2v+1}{x} \frac{du}{dx} + k^2 u = 0$$

to the form $\frac{d^2 u}{dz^2} + z^{-(2+\frac{1}{v})} u = 0$

3. Reduce the equation

$$(1-x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + n(n+1)u = 0$$

to the form $\frac{d^2 u}{dt^2} + \frac{n(n+1)}{\cosh^2 t} u = 0$

and show that it may also be reduced to a canonical form by writing

$$(1-x^2) \frac{du}{dx} = v$$

4. If $\frac{d^2 u}{dx^2} + I(x)u = 0$ $\frac{d^2 v}{dx^2} + J(x)v = 0$

prove that the product $uv = y$ satisfies the differential equation

$$\frac{d}{dx} \left[\frac{1}{Q} \left\{ \frac{d^2 y}{dx^2} + 2P \frac{dy}{dx} + y \frac{dP}{dx} \right\} \right] + Qy = 0$$

where $P = I + J$, $Q = I - J$ (Schafheitlin).

§ 31. *Equations reducible to linear equations with constant coefficients.* If we transform the equation

$$a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x) u = f(x)$$

by a substitution of type $t = \varphi(x)$, it becomes

$$a\varphi'^2 \frac{d^2 u}{dt^2} + (a\varphi'' + b\varphi') \frac{du}{dt} + cu = f(x)$$

where $\varphi'' = \frac{d^2 \varphi}{dx^2}$, $\varphi' = \frac{d\varphi}{dx}$. The new equation can be expressed as a linear equation with constant coefficients if

$$a\varphi'^2 = \lambda c \quad a\varphi'' + b\varphi' = \mu c$$

where λ and μ are constants. Eliminating ϕ' , we obtain as a necessary and sufficient condition

$$a \frac{d}{dx} \left(\sqrt{\frac{\lambda c}{a}} \right) + b \sqrt{\frac{\lambda c}{a}} = \mu c$$

This may be written in the form

$$\frac{\left(a \frac{dc}{dx} - c \frac{da}{dx} + 2bc \right)^2}{ac^3} = \text{const.}$$

The condition is evidently satisfied in the case of the equation

$$px^2 \frac{d^2u}{dx^2} + qx \frac{du}{dx} + ru = f(x)$$

where p, q, r are constants, and in this case if we take $\lambda = p$ we have $\phi'(x) = \frac{1}{x}$, and so the appropriate substitution is $t = \log x$ or $x = e^t$.

This substitution can be used to reduce the more general equation

$$a_0 x^n \frac{d^n u}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \dots a_{n-1} x \frac{du}{dx} + a_n u = f(x)$$

to a linear equation with constant coefficients; $a_0, a_1, \dots, a_n$ being regarded as constants. In carrying out the transformation it should be noticed that

$$x \frac{du}{dx} = \frac{du}{dt}$$

$$x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} = \frac{d^2u}{dt^2}$$

Let us denote the operator $x \frac{d}{dx}$ by $\mathfrak{s}$; then an expression is required for $x^m \frac{d^m u}{dx^m}$ in terms of $\mathfrak{s}$. We shall prove by induction that

$$x^m \frac{d^m u}{dx^m} = (\mathfrak{s} - m + 1)(\mathfrak{s} - m + 2) \dots \mathfrak{s} \cdot u$$

Let us assume that this formula holds for a particular integral value of m and for an arbitrary function u . We now write

$$u = \frac{dv}{dx} = \frac{1}{x} \left(x \frac{dv}{dx} \right) = e^{-t} \frac{dv}{dt}$$

then, since $\mathfrak{s} = \frac{d}{dt} = D$, say, we have

$$x^m \frac{d^{m+1}v}{dx^{m+1}} = (D - m + 1)(D - m + 2) \dots D \cdot e^{-t} \frac{dv}{dt}$$

$$= e^{-t} (D - m)(D - m + 1) \dots (D - 1) \frac{dv}{dt}$$

$$\therefore x^{m+1} \frac{d^{m+1}v}{dx^{m+1}} = (D - m)(D - m + 1) \dots (D - 1) Dv$$

Replacing D by $\mathfrak{D}$, we see that if the formula holds for a particular positive integral value of m , it holds for the next greater value. Now the formula has been proved to hold for $m = 1$; consequently it holds for all positive integral values of m .

With the aid of this formula, the transformed equation may be written down at once. Thus the equation

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + n^2 u = x^m$$

becomes

$$D(D - 1)u + Du + n^2 u = e^{mt}$$

or

$$(D^2 + n^2)u = e^{mt}$$

The solution is consequently

$$u = \frac{e^{mt}}{m^2 + n^2} + A \cos nt + B \sin nt$$

where A and B are arbitrary constants. Transforming back, we get

$$u = \frac{x^m}{m^2 + n^2} + A \cos (n \log x) + B \sin (n \log x)$$

§ 32. *Reduction of an equation of Laplace's type to a canonical form.* If a, b, c, l, m, n are constants, the equation

$$(a + lx) \frac{d^2 y}{dx^2} + (b + mx) \frac{dy}{dx} + (c + nx)y = 0 \quad (1)$$

is usually called an equation of *Laplace's type*. The problem of reducing this equation to a canonical form has been discussed by Weiler,* Schlömilch,† Pochhammer§ and Bolza.‡ Before reducing the equation to a canonical form, it will be useful to express the equation in the normal form wherein the second term is missing. The normal form which seems to be most appropriate is different according as l is or is not zero. We shall find it convenient to consider in the two different cases the more general equations ||

$$\frac{d^2 y}{dx^2} + 2 \left(\frac{e}{x} + f \right) \frac{dy}{dx} + \left(\frac{p}{x^2} + \frac{2q}{x} + r \right) y = 0 \quad (2)$$

$$\frac{d^2 y}{dx^2} + 2(e + fx) \frac{dy}{dx} + (px^2 + 2qx + r)y = 0 \quad (3)$$

where e, f, p, q, r are constants. If $l \neq 0$, equation (1) can be reduced to a particular case of (2) by writing $a + lx = x'$, while if $l = 0$, equation (1) is a particular case of equation (3).

* *Crelle's Journal*, Bd. 51 (1856), p. 123.

† *Compendium der höheren Analysis*: Brunswick (1879), Vol. II. p. 523.

§ *Math. Ann.*, Bd. 38 (1891), pp. 225, 241.

‡ *Amer. Journ.*, Vol. 15 (1893).

|| The second equation was discussed by Liouville, *Journ. de l'École Polytechnique*, Cah. 21, p. 185.

If in (2) we put* $x = \kappa x'$, $y = \nu x'^{\lambda} \varepsilon^{\mu x} y'$, we get an equation of the same type as (2), but with new constants e', f', p', q', r' defined by the equations

$$\begin{aligned} e' &= e + \lambda & f' &= \kappa f + \mu & p' &= \lambda(\lambda - 1) + 2e\lambda + p \\ q' &= \kappa q + \kappa\lambda f + \mu e + \lambda\mu & r' &= \kappa^2 r + 2\kappa\mu f + \mu^2 \end{aligned}$$

It follows from these equations that

$$\begin{aligned} p' + e' - e'^2 &= p + e - e^2 = A \text{ say} \\ q' - e'f' &= \kappa(q - ef) = \kappa B \text{ say} \\ r' - f'^2 &= \kappa^2(r - f^2) = \kappa^2 C \text{ say} \end{aligned}$$

The quantities A, B, C are in this account called *invariants* of equation (2); the quantities A and $\frac{B^2}{C}$ are *absolute invariants*. We shall write $\frac{C}{B^2} = I$.

The second term in the new equation will be missing if $e' = 0$, $f' = 0$, i.e. if $\lambda = -e$, $\mu = -\kappa f$. Equation (2) is thus reduced by the substitution $x = \kappa z$, $y = z^{-e} \varepsilon^{-\kappa f z} v$ to the form

$$\frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2\kappa B}{z} + \kappa^2 C \right) v = 0$$

Hence, by a proper choice of κ , we obtain the following canonical forms:

$$\begin{aligned} \text{I. } B \neq 0, \quad C \neq 0 & \quad \frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} + 1 \right) v = 0 \\ \text{II. } B \neq 0, \quad C = 0 & \quad \frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} \right) v = 0 \\ \text{III. } B = 0, \quad C \neq 0 & \quad \frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + 1 \right) v = 0 \\ \text{IV. } B = 0, \quad C = 0 & \quad \frac{d^2 v}{dz^2} + \frac{A}{z^2} = 0 \end{aligned}$$

The values of the constants e, f, p, q, r for an equation derived from (1) by the method indicated are as follows:

$$2e = \frac{bl - am}{l^2} \quad 2f = \frac{m}{l^2} \quad p = 0 \quad q = \frac{cl - an}{l^3} \quad r = \frac{n}{l^3}$$

To reduce an equation of type (2) to the form (1), we must first of all transform it into another equation of type (2) with $p' = 0$, and to do this we must choose λ so that $\lambda(\lambda - 1) + 2e\lambda + p = 0$. Multiplying the resulting equation by x , we obtain an equation of type (1).

$$\text{The equation} \quad \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2} \right) y = 0$$

is one of the most important equations of type (2); it is usually called *Bessel's equation*. To reduce it to the form (1), we must

* ε is used here to denote the base for natural logarithms.

choose λ so that $\lambda^2 - n^2 = 0$. An appropriate substitution is thus $y = x^n z$, and the resulting equation is

$$\frac{d^2 z}{dx^2} + \frac{2n+1}{x} \frac{dz}{dx} + z = 0$$

The differential equation (3) is transformed by the substitution

$$x = \kappa x' + \lambda \quad y = e^{\mu x' + \nu x^2} y' \quad \kappa' \neq 0 \quad (4)$$

into an equation of the same type, but with new constants e', f', p', q', r' given by the equations

$$\begin{aligned} e' &= \kappa(e + f\lambda) + \mu & f' &= \kappa^2 f + \nu & p' &= \kappa^4 p + 2\nu\kappa^2 f + \nu^2, \\ q' &= \kappa^3(q + \lambda p) + \mu\kappa^2 f + \nu\kappa(e + f\lambda) + \mu\nu \\ r' &= \kappa^2(p\lambda^2 + 2q\lambda + r) + 2\mu\kappa(p\lambda + q) + \mu^2 + \nu \end{aligned}$$

Let $P = p - f^2$ $Q = q - ef$ $R = r - e^2 - f$ $S = PR - Q^2$ then it is easy to see that if P', Q', R', S' are defined in a similar way,

$$\begin{aligned} P'x'^2 + 2Q'x' + R' &= \kappa^2(Px^2 + 2Qx + R), \\ P' &= \kappa^4 P & S' &= \kappa^6 S \end{aligned}$$

To reduce equation (3) to the normal form, we must choose $\lambda, \mu, \nu, \kappa$ so that $e' = 0, f' = 0$. Dropping the dashes, we find that the new equation is

$$\frac{d^2 y}{dx^2} + (Px^2 + 2Qx + R)y = 0$$

Now put $x = \alpha z + \beta$; then by a proper choice of α and β we can obtain the following normal forms:

$$\begin{aligned} \text{I. } P &\neq 0 & \frac{d^2 y}{dz^2} + (z^2 + \sqrt{I})y &= 0 & I &= \frac{S^2}{P^3} \\ \text{II. } P &= 0, S \neq 0 & \frac{d^2 y}{dz^2} - \frac{1}{3}zy &= 0 & \left(\begin{array}{l} \text{Scherk-Lobatto} \\ \text{equation} \end{array} \right) \\ \text{III. } P &= 0, S = 0 & \frac{d^2 y}{dz^2} + y &= 0, & \text{if } R \neq 0 \\ & & P = 0, Q = 0, R = 0 & & \frac{d^2 y}{dz^2} = 0 \end{aligned}$$

An equation of type (3) can be reduced to an equation of type (1) by choosing ν in the substitution (4) so that $p' = 0$.

We shall now show that equation (1) can in all cases be reduced to the canonical form

$$s \frac{d^2 u}{ds^2} + (\gamma - s) \frac{du}{ds} - \alpha u = 0$$

where α and γ are constants. This is Weiler's canonical form.

1°. If $l \neq 0, m \neq 0$, we write $y = e^{\lambda x} \eta$; the new equation is then

$$\begin{aligned} (a + lx) \frac{d^2 \eta}{dx^2} + [2\lambda a + b + x(2\lambda l + m)] \frac{d\eta}{dx} \\ + [\lambda^2 a + b\lambda + c + x(\lambda^2 + m\lambda + n)] \eta = 0 \end{aligned}$$

Let us choose λ so that $\lambda^2 + m\lambda + n = 0$; if this equation has two unequal roots, the new differential equation is of the form

$$(a + lx) \frac{d^2\eta}{dx^2} + (h + kx) \frac{d\eta}{dx} + j\eta = 0$$

The substitution $a + lx = v\xi$ reduces this to the form

$$\xi \frac{d^2\eta}{d\xi^2} + \left(\frac{lh - ak}{l^2} + \frac{kv}{l^2} \xi \right) \frac{d\eta}{d\xi} + \frac{jv}{l^2} \eta = 0$$

Choosing v so that $kv = -l^2$, we obtain an equation of the required type. If a, b, c, l, m, n, x, y are real, the quantities ξ, η are certainly real when the roots of $\lambda^2 + m\lambda + n = 0$ are real.

When the equation $\lambda^2 + m\lambda + n = 0$ has equal roots, $2\lambda + m = 0$, and the equation for η is of the form

$$(a + lx) \frac{d^2\eta}{dx^2} + h \frac{d\eta}{dx} + k\eta = 0$$

Putting $a + lx = v\xi^2$, we get

$$\xi \frac{d^2\eta}{d\xi^2} + \left(\frac{2h}{l} - 1 \right) \frac{d\eta}{d\xi} + \frac{4vk}{l^2} \xi \eta = 0$$

The substitution $\eta = \varepsilon^{\frac{1}{2}\xi} u$ now gives

$$\xi \frac{d^2u}{d\xi^2} + \left(\frac{2h}{l} - 1 + \xi \right) \frac{du}{d\xi} + \left[\frac{h}{l} - \frac{1}{2} + \left(\frac{4vk}{l^2} + \frac{1}{4} \right) \xi \right] u = 0$$

Choosing v so that $16vk + l^2 = 0$ and writing $\xi = -s$, we obtain an equation of the required form. If a, b, c, l, m, n, x, y are real, ξ and η are real when

$$\frac{a + lx}{am^2 - 2blm + 4cl^2}$$

is positive.

2°. $l = 0, m \neq 0$. Put $y = \eta \varepsilon^{-\frac{nx}{m}}$; the equation for η is then of the form

$$\frac{d^2\eta}{dx^2} + (h + kx) \frac{d\eta}{dx} + g\eta = 0$$

where $h = \frac{b}{a} - 2\frac{n}{m}$ $k = \frac{m}{a}$ $g = \frac{c}{a} - \frac{bn}{am} + \frac{n^2}{m^2}$

Now put $h + kx = \sqrt{2k\xi}$; we then get

$$\xi \frac{d^2\eta}{d\xi^2} + \left(\xi + \frac{1}{2} \right) \frac{d\eta}{d\xi} + \omega\eta$$

where $\omega = \frac{g}{2k} = \frac{c}{2m} - \frac{bn}{2m^2} + \frac{an^2}{2m^3}$

Writing $\xi = -s$, we obtain an equation of the required type. In this case, if a, b, c, m, n, x, y are real, s, η, ω are also real.

3°. $l=0$, $m=0$. We may get rid of the second term by writing $y = \eta e^{-\frac{bx}{2a}}$; the new equation is then

$$\frac{d^2\eta}{dx^2} + (h + kx)\eta = 0$$

where
$$h = \frac{c}{a} - \frac{b^2}{4a^2} \quad k = \frac{n}{a}$$

Now write $h + kx = v\xi^{\frac{1}{3}}$ where v is a constant; the new equation is then

$$\xi \frac{d^2\eta}{d\xi^2} + \frac{1}{3} \frac{d\eta}{d\xi} + \lambda \xi \eta = 0$$

where $\lambda = \frac{4}{9} \frac{v^3}{x^3}$. Making the further substitution $\eta = \varepsilon^{\rho\xi} u$, where ρ is a constant, we obtain

$$\xi \frac{d^2u}{d\xi^2} + \left(2\rho\xi + \frac{1}{3}\right) \frac{du}{d\xi} + \left[(\rho^2 + \lambda)\xi + \frac{1}{3}\rho\right] u = 0$$

Choosing ρ and v so that $2\rho = -1$, $\rho^2 + \lambda = 0$, we obtain

$$\xi \frac{d^2u}{d\xi^2} + \left(\frac{1}{3} - \xi\right) \frac{du}{d\xi} - \frac{1}{6} u = 0$$

an equation of the required form. It should be noticed that if a, b, c, n, x, y are real, ξ and u are real, provided $x + \frac{h}{x}$ is positive, i.e. provided

$$x > \frac{b^2 - 4ac}{4an}$$

EXAMPLES.

1. Prove that the normal form of the equation

$$s \frac{d^2u}{ds^2} + (\gamma - s) \frac{du}{ds} - \alpha u = 0$$

is
$$\frac{d^2v}{dz^2} + \frac{1}{4} \left[\frac{\gamma(2-\gamma)}{z^2} + \frac{8}{z} - \frac{1}{(2\alpha + \gamma)^2} \right] v = 0$$

Hence show that the equation

$$\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} + I \right) v = 0$$

can be reduced to Weiler's form, wherein the constants are

$$\gamma = 1 + \sqrt{1 - 4A} \quad \alpha = \frac{1}{4} + \frac{1}{4} \sqrt{1 - 4A} - \frac{1}{\sqrt{-I}} \quad (\text{O. Bolza})$$

2. If in equation (2) $(AC - B^2)^2 + B^2C = 0$, the differential equation can be reduced to the form

$$\frac{d^2y'}{dx'^2} + 2 \left(\frac{a'}{x'} + b' \right) \frac{dy'}{dx'} = 0$$

3. If in equation (3) $P^2 + S^2 = 0$, the differential equation can be reduced to the form given in the last example.

4. Prove that if an equation is of Laplace's type, its adjoint equation is also of Laplace's type.

5. Solve the equation

$$(a + lx) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0$$

by differentiation.

§ 33. *Homogeneous equations of the second type.* Let a number be associated with each of the quantities $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots$ according to the following scheme :

$$\begin{array}{ccccccc} x & y & y' & y'' & y''' & \dots \\ m & n & n-m & n-2m & n-3m & \dots \end{array}$$

and let the number associated with a product, such as $x^2y \frac{d^2y}{dx^2}$, be defined as the sum of the numbers associated with the different factors. This number will be called the *weight* of the product.

With this understanding, a differential equation $f(x, y, y', y'', \dots) = 0$ is said to be *homogeneous* when the numbers m, n can be chosen in such a way that f is the sum of a number of terms each of which has the same weight. Sometimes a differential equation is homogeneous for all values of m and n , but usually there is at most only one value of the ratio $\frac{n}{m}$ for which an equation is homogeneous ; this ratio will be called the *grade* of the equation, and will be denoted by p .

$$x^2 \left(\frac{dy}{dx} \right)^2 + x^2y \frac{d^2y}{dx^2} - y^2 = 0$$

is homogeneous for all values of m and n .

$$x^4 \frac{d^2y}{dx^2} - (x^3 + 2xy) \frac{dy}{dx} - 4y^2 = 0$$

is homogeneous when $n = 2m$, for the weights of the different terms are $n + 2m, n + 2m, 2n, 2n$ respectively. The grade is in this case 2.

$$y \frac{d^2y}{dx^2} - y^2 = 0 \text{ is homogeneous when } m = 0$$

When $m \neq 0$ it is useful to make the substitution

$$y = x^p \xi \quad \frac{dy}{dx} = x^{p-1} \eta$$

taking ξ and η as new variables. We easily find that

$$\begin{aligned} \frac{d^2y}{dx^2} &= x^{p-2} \left[(p-1)\eta + (\eta - p\xi) \frac{d\eta}{d\xi} \right] \\ \frac{d^3y}{dx^3} &= x^{p-3} \left[(p-1)(p-2)\eta + (p-3)(\eta - p\xi) \frac{d\eta}{d\xi} \right. \\ &\quad \left. + (\eta - p\xi) \left(\frac{d\eta}{d\xi} \right)^2 + (\eta - p\xi)^2 \frac{d^2\eta}{d\xi^2} \right] \\ &\dots \dots \dots \end{aligned}$$

When these expressions are substituted in the differential equation, the same power of x occurs as a factor of each term. Removing this factor, we are left with a differential equation whose order is one less than that of the original equation.

If the original equation is homogeneous for all values of m and n , it must have the form

$$F \left[y, x \frac{dy}{dx}, x^2 \frac{d^2y}{dx^2}, x^3 \frac{d^3y}{dx^3}, \dots \right] = 0$$

where F is a homogeneous function of its arguments. If the equation is of the second order, it is transformed by the above substitution into a homogeneous equation of the first order, which may be solved by the method of § 21. If the equation is of the third order, it is transformed into a homogeneous equation of the second order of grade 1. When $m = 0$, the preceding method fails. In this case we make the substitution

$$y = e^{\int u dx}$$

and take u as new dependent variable. The resulting equation is of order $s - 1$ when the original equation is of order s . A differential equation which is homogeneous for all values of m and n may be treated by this method.

EXAMPLES.

1. Solve the equations

$$x^2 y \frac{d^2 y}{dx^2} = x^2 \left(\frac{dy}{dx} \right)^2 - y^2$$

$$x^4 \frac{d^2 y}{dx^2} = \left(y - x \frac{dy}{dx} \right)^2$$

$$f(x) \frac{d^2 y}{dx^2} = F \left[x \frac{dy}{dx} - y \right]$$

§ 34. *The principle of duality and the theory of contact transformations.* A relation between three variables x, y, p , such as

$$\varphi(x, y, p) = 0$$

becomes a differential equation of the first order when the relation $dy - p dx = 0$ is satisfied. If now we make a transformation of variables,

$$x = f_1(X, Y, P) \quad y = f_2(X, Y, P) \quad p = f_3(X, Y, P) \quad . \quad (1)$$

the transformed relation $\Phi(X, Y, P) = 0$ will be a differential equation of the first order at the same time as $\varphi(x, y, p) = 0$ if the relation $dY - P dX = 0$ is a consequence of $dy - p dx = 0$. This condition is certainly satisfied if the transformation (1) is of such a nature that

$$dY - P dX = \rho(dy - p dx) \quad . \quad . \quad . \quad (2)$$

where ρ is a quantity which is neither zero nor infinite except perhaps

for certain exceptional values of x, y, p . Such a transformation is called a *contact transformation*.

Since $dy - p dx = d(y - px) + x dp$
we can evidently satisfy (2) by writing

$$Y = px - y \quad X = p \quad P = x \quad PX - Y = y. \quad (3)$$

Hence a differential equation $\varphi(x, y, p) = 0$ is transformed by this substitution into another differential equation

$$\varphi(P, PX - Y, X) = 0$$

If the second equation can be solved, the solution of the original equation can generally be deduced by a process of elimination. Let us suppose, for instance, that an integral of the second equation is given by

$$f(X, Y) = 0. \quad (4)$$

Then we have

$$\frac{\partial f}{\partial X} + P \frac{\partial f}{\partial Y} = 0$$

that is

$$x \frac{\partial f}{\partial Y} + \frac{\partial f}{\partial X} = 0. \quad (5)$$

$$\text{also} \quad -y \frac{\partial f}{\partial Y} = (Y - px) \frac{\partial f}{\partial Y} = Y \frac{\partial f}{\partial Y} + X \frac{\partial f}{\partial X} \quad (6)$$

Eliminating X and Y from equations (4), (5), (6), a relation of type $F(x, y) = 0$, which satisfies $\varphi(x, y, p) = 0$, is obtained.

It should be noticed that the transformation from x, y, p to X, Y, P is of a reciprocal nature; it is, in fact, equivalent to a reciprocation with regard to the parabola $x^2 - 2y = 0$.

Let a curve C be drawn; a tangent to it at x, y is

$$y_0 - y = p(x_0 - x)$$

where x_0 and y_0 are current coordinates. The polar of (X, Y) with regard to the parabola is, moreover,

$$Xx_0 = Y + y_0$$

The two lines are the same if $X = p, Y = px - y$.

The locus of (X, Y) is a curve C' , the polar reciprocal of C ; and C is known to be the polar reciprocal of C' , that is, the locus of the poles of the tangents to C' , taken with respect to the parabola. Thus the point x, y on C is the pole of the tangent to C' at the point X, Y ; by a similar process, we should find

$$x = P \quad y = PX - Y$$

It is useful to regard the three variables x, y, p as the coordinates of a linear element, that is, a point and a direction through the point.* The condition $dy - p dx = 0$ then states that if the point (x, y) describes a curve, the direction of the linear element is that of the tangent to the curve; it is, in fact, the condition that a

*The quantity p represents the tangent of the angle which this direction makes with the axis of x . Notice that in this specification of the element no distinction is drawn between a direction and its opposite.

succession of consecutive linear elements should form a curve. The invariance of this condition under the transformation also implies that if two curves touch, their corresponding curves touch, whence the name *contact transformation*. For a full development of this theory the reader is referred to Sophus Lie's *Geometrie der Berührungstransformationen*, Leipzig, 1896, and to A. Cohen's *The Lie Theory of One-parameter Groups* (1911).

The reciprocal transformation (3) may also be used to transform certain differential equations of the second order; we evidently have

$$dP - R dX = dx - R dp = -R \left(-dp - \frac{1}{R} dx \right)$$

$$\text{Hence, if } R = \frac{dP}{dX} = \frac{d^2Y}{dX^2} \quad \frac{1}{R} = \frac{dp}{dx} = \frac{d^2y}{dx^2}$$

the second derivatives are consequently connected by the relation

$$\frac{d^2Y}{dX^2} \cdot \frac{d^2y}{dx^2} = 1$$

EXAMPLES.

1. Prove that the following transformations are contact transformations :

$$X = x + \frac{ap}{\sqrt{1+p^2}} \quad Y = y - \frac{a}{\sqrt{1+p^2}} \quad P = p$$

$$X = x - yp \quad Y = y\sqrt{p^2 - 1} \quad P = -\frac{p}{\sqrt{p^2 - 1}}$$

$$X = -\frac{p}{\sqrt{p^2 - 1}} \quad Y = \frac{y - px}{\sqrt{p^2 - 1}} \quad P = x - yp$$

2. Solve the equations

$$\begin{aligned} x(y - px) &= y \\ p(x - yp) &= y(p^2 - 1) \end{aligned}$$

3. If $\phi(x, y, p)$ is an integrating factor of the expression $f(x, y, p, r)$, then $R\phi(P, PX - Y, X)$ is an integrating factor of the transformed equation $f(P, PX - Y, X, R^{-1})$.

§ 35. *Jacobi's equation.* An interesting type of differential equation occurs in the geometrical theory of the linear transformation

$$x_1 = \frac{l_1x + l_2y + l}{n_1x + n_2y + n} \quad y_1 = \frac{m_1x + m_2y + m}{n_1x + n_2y + n} \quad (1)$$

wherein $l_1, l_2, l, m_1, m_2, m, n_1, n_2, n$ are constants. This transformation associates a point P_1 with P in such a way that if the point P describes a straight line, P_1 also describes a straight line. Let us now endeavour to find a curve C such that the tangent at any

point P passes through the corresponding point P_1 . Since the equation of the tangent at P is

$$y_0 - y = \frac{dy}{dx}(x_0 - x)$$

where (x_0, y_0) are current coordinates, we are led to the differential equation

$$(m_1x + m_2y + m)dx - (l_1x + l_2y + l)dy - (n_1x + n_2y + n)(ydx - xdy) = 0 \quad (2)$$

This differential equation is transformed into itself by the substitution (1), for the curve C corresponds in the transformation to a curve C_1 passing through P_1 , the tangent at P to C corresponds to the tangent at P_1 to C_1 ; and since the tangent at P passes through P_1 , the tangent at P_1 will pass through the point P_2 corresponding to P_1 . Hence the curve C_1 also satisfies the requirements, and so the differential equation for C_1 must have the same form in the coordinates (x_1, y_1) as equation (2).

This result is easily verified analytically. It is important, moreover, to notice that if we make an arbitrary linear transformation

$$X = \frac{p + p_1x + p_2y}{r + r_1x + r_2y} \quad Y = \frac{q + q_1x + q_2y}{r + r_1x + r_2y} \quad (3)$$

wherein $p_1, p_2, p, q_1, q_2, q, r_1, r_2, r$ are constants, the differential equation (2) is transformed into a similar equation with a different set of constants. In fact, since the tangent at a point (x, y) of any curve γ is transformed by the substitution (3) into the tangent at (X, Y) to the corresponding curve Γ , we must have a relation of type

$$Y - Y_0 - \frac{dY}{dX}(X - X_0) = \mu \left[y - y_0 - \frac{dy}{dx}(x - x_0) \right] \quad (4)$$

where (X_0, Y_0) corresponds to (x_0, y_0) and μ is some function of x, y, x_0, y_0 , which is different from zero. Substituting the values of X_0, Y_0 in terms of x_0, y_0 given by (3) and equating coefficients, we find that

$$\left. \begin{aligned} & \frac{dx}{q_2dX - p_2dY - r_2(YdX - XdY)} \\ &= \frac{dy}{p_1dY - q_1dX + r_1(YdX - XdY)} \\ &= \frac{ydx - xdy}{p dY - q dX + r(YdX - XdY)} \end{aligned} \right\} \dots \quad (5)$$

Substituting for x, y, dx, dy in (2), a differential equation of the same type as (2) is obtained. When the substitution (3) is the same as (1), the differential equation (2) may be written in the form

$$Ydx - Xdy = ydx - xdy$$

and it follows at once from (5) that this is equivalent to an equation exactly like (2), but in the coordinates (X, Y) .

We shall now endeavour to choose the substitution (3) so that equation (2) is reduced to a simple form. It is clear that the substitution (1), under which (2) is invariant, will be transformed into the substitution under which the new equation is invariant. Let us choose the transformation (3) so that the substitution corresponding to (1) is simply

$$X_1 = \frac{\lambda X}{\nu} \quad Y_1 = \frac{\mu Y}{\nu}$$

where λ, μ, ν are constants; then the new differential equation is

$$\mu Y dX - \lambda X dY - \nu(Y dX - X dY) = 0$$

or
$$(\mu - \nu) \frac{dX}{X} + (\nu - \lambda) \frac{dY}{Y} = 0$$

The solution is thus of the form $X^{\mu-\nu} Y^{\nu-\lambda} = \text{const.}$ or

$$(p + p_1x + p_2y)^{\mu-\nu} (q + q_1x + q_2y)^{\nu-\lambda} (r + r_1x + r_2y)^{\lambda-\mu} = \text{const.} \quad (6)$$

To determine the constants p, q, r , etc., we must identify the substitution

$$\begin{aligned} \frac{p + p_1x_1 + p_2y_1}{r + r_1x_1 + r_2y_1} &= \frac{\lambda p + p_1x + p_2y}{\nu r + r_1x + r_2y} \\ \frac{q + q_1x_1 + q_2y_1}{r + r_1x_1 + r_2y_1} &= \frac{\mu q + q_1x + q_2y}{\nu r + r_1x + r_2y} \end{aligned}$$

with (1). This gives

$$\begin{aligned} p_1(l_1x + l_2y + l) + p_2(m_1x + m_2y + m) + p(n_1x + n_2y + n) &\equiv \theta\lambda(p_1x + p_2y + p) \\ q_1(l_1x + l_2y + l) + q_2(m_1x + m_2y + m) + q(n_1x + n_2y + n) &\equiv \theta\mu(q_1x + q_2y + q) \\ r_1(l_1x + l_2y + l) + r_2(m_1x + m_2y + m) + r(n_1x + n_2y + n) &\equiv \theta\nu(r_1x + r_2y + r) \end{aligned}$$

where θ is some constant. The first equation gives

$$\begin{aligned} p_1(l_1 - \theta\lambda) + p_2m_1 + pn_1 &= 0 \\ p_1l_2 + p_2(m_2 - \theta\lambda) + pn_2 &= 0 \\ p_1l + p_2m + p(n - \theta\lambda) &= 0 \end{aligned}$$

hence $\theta\lambda$ must be a root of the equation in σ ,

$$\begin{vmatrix} l_1 - \sigma & m_1 & n_1 \\ l_2 & m_2 - \sigma & n_2 \\ l & m & n - \sigma \end{vmatrix} = 0 \quad (7)$$

and in the same way it can be shown that $\theta\mu$ and $\theta\nu$ are roots of this equation. Since we require only the ratios of λ, μ, ν , we may without loss of generality take $\theta = 1$.

The case in which two roots of equation (7) are equal has been discussed by J. A. Serret; we may deduce the result from the previous one by writing $\lambda = \mu + \varepsilon$ and making $\varepsilon \rightarrow 0$.

Taking logarithms of both sides of (6) and using the values

$$\begin{aligned} p_1 &= m_1 n_2 - m_2 n_1 + n_1 \lambda & p_2 &= n_1 l_2 - n_2 l_1 + n_2 \lambda \\ p &= l_1 m_2 - l_2 m_1 - \lambda(l_1 + m_2) + \lambda^2 \end{aligned}$$

we find that the coefficient of ε on the left-hand side is

$$\frac{\mu - \nu}{q + q_1 x + q_2 y} (n_1 x + n_2 y + 2\mu - l_1 - m_2) - \log(q + q_1 x + q_2 y) + \log(r + r_1 x + r_2 y)$$

Equating this to a constant, we obtain the solution

$$\frac{r + r_1 x + r_2 y}{q + q_1 x + q_2 y} \exp \left\{ (\mu - \nu) \frac{n_1 x + n_2 y + 2\mu - l_1 - m_2}{q + q_1 x + q_2 y} \right\} = \text{const.}$$

EXAMPLES.

1. Solve the equations

$$\begin{aligned} (x + 1) dx - (3y - 1) dy - (5x - 9y + 6)(y dx - x dy) &= 0 \\ (x + 1) dx - (3y - 1) dy - (7x - 9y + 8)(y dx - x dy) &= 0 \end{aligned}$$

2. Prove that if the transformation (1) leaves the angle between any two lines unaltered, the curve C is an equiangular spiral. Consider the case in which the transformation consists of a rotation round a point O followed by a magnification about this point.

After finishing this chapter the reader will find it worth while to try some of the miscellaneous examples at the end of the book.

CHAPTER IV

GEOMETRICAL APPLICATIONS

§ 36. *The differential equation of a family of algebraic curves.* Let us consider a family of algebraic curves defined by the equation

$$f(x, y, c) = 0 \dots \dots \dots (1)$$

wherein c is an arbitrary parameter. At a point (x, y) of one of these curves, the direction of the tangent is specified by the value of $p = \frac{dy}{dx}$, and this is given by the equation

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial y} = 0 \dots \dots \dots (2)$$

We shall suppose that f is a single-valued function of its arguments, so that the last equation determines just one value of p when x, y and c are given ; it fails to determine p , however, when $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero.

If the parameter c be eliminated between (1) and (2), a differential equation of type $\phi(x, y, p) = 0 \dots \dots \dots (3)$ is obtained, and this is regarded as the differential equation of the family of curves represented by equation (1).

If f is a polynomial of the n^{th} degree in c , there will generally be n values of c when the point (x, y) is known ; consequently there are generally n real or imaginary curves of type (1) through a point (x, y) , and so there are generally n real or imaginary values of p , some of which may be equal. We can then expect ϕ to be a polynomial of the n^{th} degree in p , whose coefficients are rational functions of x and y . It is not true, however, that every differential equation of the n^{th} degree with rational coefficients corresponds in this way to a family of algebraic curves ; for instance, the linear equation

$$\frac{dy}{dx} = xy$$

corresponds to the family $y = ce^{\frac{1}{2}x^2}$, which is not algebraic.

The differential equations of the n^{th} order with rational coefficients which correspond to families of algebraic curves are of a special type, but they are of considerable interest, because the

differential equation can be used to define the family of curves, and certain loci connected with the family may be derived at once from the differential equation without finding its general solution.

EXAMPLES.

1. Find the differential equation of the family of circles

$$(x - c)^2 + y^2 = 1$$

We have the relation $2(x - c) + 2yp = 0$;

$$\therefore y^2(1 + p^2) = 1$$

2. Find the differential equations of the following families of curves :

$$y^2 - c^2 = x - c \qquad (y - c)^2 = x^2 \qquad y^2 = (x - c)^2$$

$$\frac{x^2}{\alpha + c} + \frac{y^2}{\beta + c} = 1 \qquad y^2 + c^2 x^2 = 3cxy$$

3. Prove that the points of contact of the tangents from the point (x_0, y_0) to the curves of the family $\varphi(x, y, p) = 0$ lie on the curve

$$\varphi\left(x, y, \frac{y - y_0}{x - x_0}\right) = 0 \quad (\text{M. Fouret})$$

§ 37. *Singular solutions of differential equations.* Let us consider a relation of the form $\varphi(x, y, p) = 0$

where φ is a polynomial of the n^{th} degree in p , the coefficients being rational functions of x and y . If x, y, p are regarded as the coordinates of a linear element, p being the tangent of the angle which the element makes with the axis of x , the above equation tells us that there are generally n linear elements for each point P whose coordinates are x, y .

A curve which is such that the tangent at each point of it has the same direction as one of the linear elements associated with this point is obtained by putting $p = \frac{dy}{dx}$ in the relation $\varphi(x, y, p) = 0$ and solving the resulting differential equation. Such a curve is called an *integral curve* of the differential equation. It seems natural to expect that there will be n real or imaginary integral curves through a point (x, y) for which the n values of p are all distinct, for when p is known the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$ are determined uniquely by the equations

$$\begin{aligned} \frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} \frac{\partial \varphi}{\partial p} + \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} p &= 0 \\ \frac{d^3y}{dx^3} \frac{\partial \varphi}{\partial p} + \frac{\partial^2 \varphi}{\partial x^2} + 2p \frac{\partial^2 \varphi}{\partial x \partial y} + p^2 \frac{\partial^2 \varphi}{\partial y^2} + 2 \frac{d^2y}{dx^2} \frac{\partial^2 \varphi}{\partial x \partial p} \\ + \left(\frac{d^2y}{dx^2}\right)^2 \frac{\partial^2 \varphi}{\partial p^2} + 2p \frac{d^2y}{dx^2} \frac{\partial^2 \varphi}{\partial y \partial p} &= 0 \\ . & \end{aligned}$$

We are not justified, however, in jumping to the conclusion that the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$ which are associated in this way with the value of x are sufficient to define uniquely a solution of the differential equation; indeed, it is not certain from *a priori* considerations that they define a function at all. Points of this kind are usually settled by the so-called existence theorems for differential equations which are too difficult to be considered at this stage. At present we shall be concerned only with the case in which the equations (2) fail to determine the values of $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$, when x, y and p are given. This occurs when $\frac{\partial \phi}{\partial p} = 0$ in virtue of the value which has been given to p , that is, a value determined by the equation

$$\phi(x, y, p) = 0$$

Now the equations $\phi = 0, \frac{\partial \phi}{\partial p} = 0$ imply that for the chosen values of x and y the equation $\phi = 0$, when considered as an equation for p , has equal roots. This means that either two distinct integral curves have the same tangent at (x, y) or that a single integral curve has a cusp or multiple point with two or more coincident tangents at the point (x, y) . In the first case the two integral curves may or may not be consecutive.

If a family of integral curves is represented by the equation $f(x, y, c) = 0$, there will frequently be an envelope, *i.e.* a curve which touches each member of the family and a tac-locus, *i.e.* a locus of points at which two curves of the family touch one another (Fig. 4A).

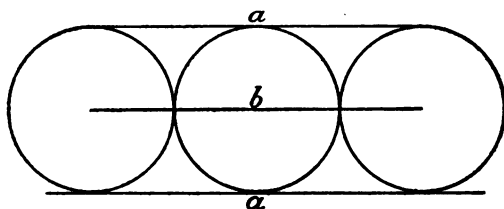


FIG. 4A.—System of equal circles with their centres on a line b . The envelope consists of the lines a and the tac-locus of the line b .

The equation of the envelope will also provide us with a solution of the differential equation, for if the envelope touches an integral curve of the family $f = 0$ at a point (x, y) , the value of p for the envelope is the same as for the integral curve $f = 0$, and consequently the equation $\phi(x, y, p) = 0$ is satisfied.

The equation of the envelope of the family $f(x, y, c) = 0$ cannot generally be obtained by giving a particular value to c ; consequently if $f(x, y, c) = 0$ is regarded as the general solution of the differential equation $\phi(x, y, p) = 0$, the equation of the envelope must be

regarded as a *singular solution* of the differential equation. This result is due to Lagrange. The envelope will be regarded as a *singular integral curve*.

Both the envelope and the tac-locus of curves belonging to the family $f(x, y, c) = 0$ fulfil the requirement of being the locus of points at which two distinct integral curves touch, but whereas the envelope is itself an integral curve the tac-locus is not necessarily so, for there is no reason why the tangent at a point (x, y) of the tac-locus should coincide with a tangent to one of the integral curves through this point. Hence the tac-locus does not generally provide us with a solution of the differential equation. For a similar reason the cusp-locus does not generally provide us with a solution of the differential equation.

If we eliminate p from the equations $\varphi = 0$, $\frac{\partial \varphi}{\partial p} = 0$, we shall obtain a relation between x and y which includes the equations of the envelope, tac-locus and cusp-locus when they exist. In performing the elimination, care must be taken to see that no factor is rejected without examination. A good method is to use Sylvester's dialytic method of elimination and to call the eliminant the *p-discriminant*.

Darboux has shown that if we start from the equation

$$\varphi(x, y, p) = 0$$

there is generally no singular solution, and the *p-discriminant* simply gives the locus of cusps of integral curves.* It is easy to see, in fact, that a singular solution only exists when the function φ satisfies a certain condition, for if the *p-discriminant* satisfies the differential equation, the value of $\frac{dy}{dx}$ for the eliminant of the equations

$$\varphi(x, y, p) = 0 \quad \frac{\partial \varphi}{\partial p} = 0$$

should be equal to one of the values of p given by $\varphi = 0$. Now the value of $\frac{dy}{dx}$ is given by the equation

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{dy}{dx} = 0$$

hence, in order that a singular solution may exist, the function φ must satisfy the equation

$$\varphi \left(x, y, - \frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}} \right) = 0$$

* *Comptes Rendus*, t. 70, (1870). *Bulletin des Sciences Mathématiques* (1873), p. 158.

This equation may be satisfied either identically or in virtue of $\varphi(x, y, p) = 0$.

If we eliminate p between the last equation and $\varphi(x, y, p) = 0$; in other words, if we eliminate p from the equations

$$\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0 \quad \varphi(x, y, p) = 0$$

an equation is obtained which includes the singular solution when it exists. The equation also gives the locus of points of inflexion on integral curves, because the equation $\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0$ is satisfied when $\frac{d^2y}{dx^2} = 0$.

The first example of a singular solution was given by Leibnitz in 1694. Brook Taylor showed in 1715 that a singular solution can be deduced directly from the differential equation in the case of the equation

$$\varphi(x, y, p) \equiv (1 + x^2)p^2 - 2xyp + y^2 - 1 = 0$$

This equation is of Clairaut's type, and so its general solution is

$$f(x, y, c) \equiv (1 + x^2)c^2 - 2xyc + y^2 - 1 = 0$$

The p -discriminant is

$$(1 + x^2)(y^2 - 1) - x^2y^2 = 0$$

or

$$y^2 - x^2 = 1$$

$$\text{Also} \quad \frac{\partial \varphi}{\partial x} = 2xp^2 - 2yp \quad \frac{\partial \varphi}{\partial y} = 2y - 2xp$$

and so

$$\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0$$

Hence $y^2 - x^2 = 1$ is a solution of the differential equation, and it cannot be derived from $f(x, y, c) = 0$ by giving a special value to c ; consequently it is a singular solution.

EXAMPLES.

1. Find the singular solutions of the following differential equations :

$$\begin{aligned} \alpha^2[x + y + p(y - x)]^2 &= (xp - y)^2(x^2 + y^2) \\ p^2(2x^2 + 1) + p(x^2 + 2xy + y^2 + 2) + 2y^2 + 1 &= 0 \\ p^3 - 4xyp + 8y^2 &= 0 \end{aligned}$$

Show that in the last example $y = 0$ is included in the general integral and is also a true 'singular solution' in the sense that it is included in the envelope (Cauchy).

2. If a differential equation $\varphi(x, y, p) = 0$ is transformed into $\Phi(X, Y, P) = 0$ by the substitution $x = P$, $y = PX - Y$, $p = X$, the p -discriminant is transformed into the locus given by the equations $\Phi = 0$, $\frac{\partial \Phi}{\partial X} + P \frac{\partial \Phi}{\partial Y} = 0$.

§ 38. *The c-discriminant.* The envelope of a family of integral curves $f(x, y, c) = 0$ may also be obtained by forming the c -discriminant, i.e. by eliminating c from the equations

$$f(x, y, c) = 0 \quad \frac{\partial}{\partial x} f(x, y, c) = 0 \quad . \quad . \quad . \quad (1)$$

To see this, let us suppose that (x, y) is a point for which the two equations can be satisfied, and that an appropriate value of c , depending on x and y , has been derived from these equations. Call this value of c , $c(x, y)$; then, if we calculate $\frac{dy}{dx}$ from the equation

$$f[x, y, c(x, y)] = 0 \quad . \quad . \quad . \quad . \quad (2)$$

we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial c} \left[\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y} \frac{dy}{dx} \right] = 0$$

Since $\frac{\partial f}{\partial c} = 0$, it follows that, if the quantity within square brackets is not infinite, the value of $\frac{dy}{dx}$ for the curve (2) is the same as for the curve $f(x, y, c) = 0$, in which c is regarded as constant. Hence the curve (2) touches $f(x, y, c) = 0$ at the point (x, y) , and is consequently an envelope of the family of curves $f = 0$.

If the equation $f(x, y, c) = 0$ is a polynomial of the n^{th} degree in c with single-valued coefficients, there are generally n curves of the family $f = 0$ through an arbitrary point x, y . If, however, this point is on the locus obtained by equating the c -discriminant to zero, the equation for determining c has equal roots. The envelope is the locus of the points of intersection* of consecutive curves of the family $f = 0$. It is easy to see that the c -discriminant may give other curves besides the envelope. Let us regard $z \equiv c$ as a third coordinate, so that (x, y, z) are the coordinates of a point in space.† The equation $f(x, y, c) = 0$ or $f(x, y, z) = 0$ then represents a surface S .

Now, if the values of x and y are given, the point (x, y, z) lies on a line parallel to the axis of z , and the possible values of z will be represented by the points in which this line meets the surface S . It is clear that two of the values of z will be equal either if the line touches S or if it meets a nodal or cuspidal curve on S .

Let us consider the simple case when the surface S is an algebraic surface of the m^{th} degree having an r -fold point P at infinity on the axis of z . To find the degree of the discriminant, we must find

* Chrystal has pointed out that the ultimate intersections of consecutive curves of a family $f(x, y, c) = 0$ may be the same for all curves, in which case there is no true envelope. Edinburgh, *Phil. Trans.*, Vol. 38 (1896), p. 803.

† Cf. A. Cayley, *Mess. of Math.*, Vol. 2 (1872), p. 6, Vol. 6 (1877), p. 23; M. J. M. Hill, *Proc. London Math. Soc.*, Ser. 1, Vol. 19 (1888), p. 565; R. W. H. T. Hudson, *Ibid.*, Vol. 33 (1901), p. 380.

the degree of the orthogonal projection on the plane of x, y of the surfaces $f = 0$, $\frac{\partial f}{\partial z} = 0$.

It will be sufficient to consider the sections of these surfaces by a plane perpendicular to the plane of (x, y) . The section of S is a curve of degree m having an r -fold point at P , while the section of $\frac{\partial f}{\partial z} = 0$ is the first polar of this curve for P . Now it is known that the first polar of a curve F for a point P has one intersection with F at a point of contact of a tangent from P , two intersections with F at a double point of F and three intersections with F at a cusp on F . The locus of the points of contact of tangents from P to the surface S projects into the envelope of curves of the family $f(x, y, c) = 0$, c being regarded now as a parameter.



FIG. 5.—The Node locus or locus of double points on the curves f is represented here by the line a .

The locus of double points on sections F which do not arise from contacts of the plane with the surface S , projects into the locus of double points of curves of the family $f(x, y, c) = 0$. This locus is the orthogonal projection of the double curve of S . Similarly, the locus of cusps on sections F or the cuspidal curve of S projects into the cusp-locus of the family $f(x, y, c) = 0$.

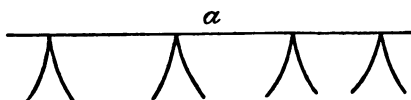


FIG. 6.—The cusp-locus is represented here by the line a .

It is now clear that the envelope appears as a simple factor of the c -discriminant, the locus of double points as a double factor and the cusp-locus as a triple factor.

It may happen that a portion of the curve of contact of the tangent cone from P to the surface S consists of a plane curve lying in a plane parallel to the plane of (x, y) . In this case the projection of the curve of contact of tangents from P consists of two parts, the true envelope of curves of the family $f(x, y, c)$ and a special curve which is either a particular curve of the family or a part of a degenerate curve of the family. Such a curve generally appears once as a factor of the discriminant, and is to be regarded as a limiting curve of the family as c approaches some special value.* Such a curve will be called a *particular curve*.

* Cf. Cohen's *Differential Equations*, p. 71. See also A. Cauchy, *Leçons sur le calcul différentiel et le calcul intégral*, Paris (1844); M. Petrovitch, *Math. Ann.*, Bd. 50 (1898), p. 103; W. Dyck, *Sitzungsberichte*, Munich (1910); P. Stäckel, *Ibid.* (1912).

§ 39. *The p-discriminant.* In discussing the p -discriminant, it will be convenient to regard $\phi(x, y, p) = 0$ as the primitive of a new differential equation obtained by eliminating the parameter p . The integral curves of the new equation are derived from the integral curves of the original equation by finding the loci of points for which p has a constant value, *i.e.* for which the tangent has a given direction.* Let us call the integral curves of the two differential equations f -curves and ϕ -curves respectively. For the sake of simplicity, we shall assume that they are algebraic.

The envelope of the f -curves is easily seen to be an envelope of the ϕ -curves. The tac-locus for the f -curves is a locus of double points on the ϕ -curves. The cusp-locus for the f -curves is generally an envelope of the ϕ -curves; but if the cuspidal tangents are all in the same direction, it becomes a particular ϕ -curve or portion of such curve. Finally, a particular f -curve is generally a cusp-locus for the ϕ -curves.

These results may be obtained geometrically by considering the surface S . A ϕ -curve is then the orthogonal projection of the curve of contact of tangents in a given direction parallel to the plane of (x, y) , *i.e.* of tangents which pass through a given point Q at infinity in the plane of (x, y) . Call this curve of contact q , and let p be the curve of contact from P , the point at infinity on the axis of z .

Now, if p meets q at a point R , the ϕ -curve which is the projection of q will generally touch the projection of p at the projection of R , *i.e.* it will touch the envelope of the f -curves. Hence the envelope of the f -curves is generally an envelope of ϕ -curves.

If, however, the tangent at R to q passes through P , the curve q will project into a ϕ -curve having a cusp at the projection of R . Now, if the tangent at R to q passes through P , RQ and RP are conjugate tangents † to the surface at R , and so the relation between

* Cf. M. J. M. Hill, *loc. cit.*; J. H. M. Wedderburn, *Proc. Roy. Soc. Edinburgh*, Vol. 24 (1902), p. 400. At Chrystal's suggestion Wedderburn calls the curves *isoclinal lines*.

This term was used by J. Massau in his work on the graphical solution of differential equations.

† Conjugate tangents to a surface at a point R may be defined as follows:

Let the point R be taken as origin of coordinates, and the tangent plane at R to the surface as plane of x, y . If R is an ordinary point of the surface, the equation of the surface can be thrown into the form

$$z = ax^2 + 2hxy + by^2 + 2fyz + 2gzx + cz^2 + \dots$$

The quadric S obtained by retaining only the first three terms on the right-hand side of this equation can be regarded as a second approximation to the surface when x and y are small, and of the same order of magnitude; notice that the terms $2fyz$, $2gzx$, cz^2 are small compared with ax^2 , $2hxy$, by^2 .

Two tangents at R to the surface are also tangents to the quadric S . They are said to be *conjugate tangents* of the R surface when they are polar lines with regard to S . This means that if we take any point P on either tangent, and draw a tangent cone from P to the quadric S , the tangent at R to the curve of contact is the tangent conjugate to RP .

Since, moreover, the quadric S is a second approximation to the surface, the curve of contact of the tangent cone from P to the surface has the same tangent at R as the R curve of contact of the tangent cone from P to the quadric.

them is a reciprocal one ; consequently the tangent at R to p passes through Q. Hence in this case the tangent at R to p is parallel to the plane of (x, y) . It is clear then that if a portion of p consists of a curve in a plane parallel to the plane of (x, y) , this curve projects into a special curve for the f 's, which is a cusp-locus for the ϕ 's.

At a point where two f 's touch, two branches of a ϕ cross, and so the tac-locus for the f -curves is a node-locus for the ϕ -curves.

Finally, it is clear that each q -curve touches the cuspidal edge of the surface S, and so the cusp-locus for the f -curves is generally an envelope for the ϕ -curves.

The number of times the envelope, tac-locus, cusp-locus and particular curve appear as factors of the p -discriminant may be found at once by regarding them as an envelope, node-locus, envelope (or particular curve) and cusp-locus respectively for the ϕ -curves when p is regarded as a parameter.

The way in which the different loci appear as factors of the c - and p -discriminants is shown in the following table :

c-discriminant.		p-discriminant.	
Envelope	1	Envelope	1
Particular curve	1	Cusp-locus	1
Node-locus	2	Tac-locus	2
Cusp-locus	3	Particular curve	3

EXAMPLES.

1. Let us consider the equation

$$\phi(x, y, p) \equiv y + \frac{2}{3}p^3 - (x + p^3) = 0 \quad . \quad . \quad . \quad (1)$$

which is of Lagrange's type. A family of integral curves is given by

$$(y - c)^3 + \frac{4}{3}(x - c)^3 = 0 \quad . \quad . \quad . \quad (2)$$

The p -discriminant is obtained by eliminating p from equation (1) and

$$2p(p - 1) = 0$$

it is

$$(y - x)(y - x - \frac{1}{3}) = 0$$

The c -discriminant is obtained by eliminating c from (2) and

$$2(y - c) + \frac{4}{3}(x - c)^3 = 0$$

We easily find that $(x - c)^3 + (x - c)^4 = 0$

Hence the c -discriminant is

$$(y - x)^3(y - x - \frac{1}{3}) = 0$$

In this case $y = x$ is a cusp-locus and $y - x = \frac{1}{3}$ the envelope. It is easy to verify that $\phi = 0$ is satisfied by $y - x = \frac{1}{3}$, $p = 1$; hence $y - x = \frac{1}{3}$ is a singular solution.

2. In the case of the equation

$$xp^2 - (x-1)^2 = 0$$

the p -discriminant is $x(x-1)^2 = 0$. A family of integral curves is given by the equation $9(y+c)^2 = 4x(x-3)^2$

and the c -discriminant is $x(x-3)^2 = 0$. In this case $x=0$ is the envelope or singular solution, $x=1$ a tac-locus and $x=3$ the node-locus.

3. Discuss the p - and c -discriminants of the equations

$$y^2(1+p^2) = 1 \quad y^2 + (x-c)^2 = 1$$

$$(x+yp)^2 + 4(y-xp)^2 = a^2(1+p^2)(x^2+y^2)$$

$$(x+yc)^{\frac{2}{3}} + (y-xc)^{\frac{2}{3}} = a^{\frac{2}{3}}(1+c^2)^{\frac{2}{3}}$$

$$(3y+2)^2 p^2 = 4(1+y) \quad (x-c)^2 = y^2 + y^3$$

The solution of each equation is given in the same line.

4. Prove that in the case of the equation

$$(2x-p)^2 + x(y-x^2)^2(2x-p) + (y-x^2)^2 = 0$$

the particular curve $y = x^2$ appears as a factor of the p -discriminant. Taking

$$y(1+cx) = x^2 + cx^3 + c^2$$

as a family of integral curves, show that $y - x^2 = 0$ belongs to this family. What is the geometrical meaning of the factor $x^2 - 4$ of the c -discriminant? (Petrovitch.)

5. If $\frac{\partial^2 f}{\partial c^2} = 0$ at all points of the node-locus of the family of curves $f(x, y, c) = 0$, the node-locus N is also an envelope and N^2 a factor of the discriminant. If $\frac{\partial^2 f}{\partial c^2} = 0$ at all points of a cusp-locus C , this is an envelope and C^4 is a factor of the discriminant. (M. J. M. Hill.)

6. Prove by the principle of duality, or otherwise, that the result of eliminating p from the equations $\varphi(x, y, p) = 0$, $\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0$ gives the envelope, tac-locus, flex-locus and particular curve.

7. A series of differential equations is derived from the equation $\varphi(x, y, p) = 0$ by the following rule. If $\varphi_n(x, y, p) = 0$ is the n^{th} equation, $\varphi_n(x, y, c) = 0$ is the general solution of the $(n+1)^{\text{th}}$ equation. Verify that the third equation is the same as the first when the differential equation is of type

$$y - G(p) + pG'(p) = F\{x + G'(p)\}$$

where F and G are arbitrary functions.

8. If a differential equation $\varphi(x, y, p) = 0$ is unaltered by the infinitesimal transformation

$$x' + \varepsilon \xi(x, y) \quad y' = y + \varepsilon \eta(x, y)$$

where ε is a small quantity, show that it possesses the singular solution

$$\varphi(x, y, \eta/\xi) = 0$$

9. The tangential equation of a pencil of curves $f(x, y) + cg(x, y) = 0$ of order n is of degree $2(n-1)$ in c and of class $n(n-1)$. The c -discriminant of this equation is of class $2n(n-1)(2n-3)$. What is the geometrical meaning of the equation obtained by equating the c -discriminant to zero? Hence find the class of the envelope of inflexional tangents of a pencil of cubics.

§ 40. *Differential equations which can be resolved into factors.* A differential equation of the first order which can be resolved into factors such as

$$[p - \psi_1(x, y)][p - \psi_2(x, y)] \dots [p - \psi_n(x, y)] = 0 \quad (1)$$

should really be regarded as n different equations and not as a single equation.

It should be noticed, however, that a differential equation of this type arises when we form the differential equation corresponding to a primitive of type

$$[c - g_1(x, y)][c - g_2(x, y)] = 0 \quad (2)$$

In fact, when we differentiate with regard to x , we get

$$c \left[\frac{\partial g_1}{\partial x} + p \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} + p \frac{\partial g_2}{\partial y} \right] \\ = \left(g_1 \frac{\partial g_2}{\partial y} + g_2 \frac{\partial g_1}{\partial y} \right) p + g_1 \frac{\partial g_2}{\partial x} + g_2 \frac{\partial g_1}{\partial x}$$

and so the corresponding differential equation is

$$(g_1 - g_2)^2 \left(\frac{\partial g_1}{\partial x} + p \frac{\partial g_1}{\partial y} \right) \left(\frac{\partial g_2}{\partial x} + p \frac{\partial g_2}{\partial y} \right) = 0 \quad (3)$$

It should be noticed that the factor $\frac{\partial g_1}{\partial x} + p \frac{\partial g_1}{\partial y} = 0$ is the differential equation corresponding to $g_1(x, y) = c$, and the factor $\frac{\partial g_2}{\partial x} + p \frac{\partial g_2}{\partial y} = 0$ the differential equation corresponding to $g_2(x, y) = c$.

In a similar way, if the primitive of the equations

$$p = \psi_1(x, y) \quad p = \psi_2(x, y), \dots p = \psi_n(x, y) \quad (4)$$

can be put respectively in the forms

$$g_1(x, y) = c_1 \quad g_2(x, y) = c_2, \dots g_n(x, y) = c_n$$

the result obtained by substituting the values of $c_1, c_2, \dots c_n$ in the equation

$$(c - c_1)(c - c_2) \dots (c - c_n) = 0$$

can be regarded as a primitive of (1). We thus obtain the solution

$$(c - g_1)(c - g_2) \dots (c - g_n) = 0 \quad (5)$$

The p -discriminant of equation (1) is

$$(\psi_1 - \psi_2)^2 (\psi_1 - \psi_3)^2 \dots (\psi_1 - \psi_n)^2 (\psi_2 - \psi_3)^2 \dots (\psi_2 - \psi_n)^2 \dots \\ (\psi_{n-1} - \psi_n)^2 = 0$$

and evidently represents a tac-locus of the curves given by (5), for

the equation $\psi_r = \psi_s$ implies that a curve of type $g_r = c_r$ has the same tangent as a curve of type $g_s = c_s$. Notice that the factor $(\psi_r - \psi_s)$ appears twice in the discriminant, as it should do.

The c -discriminant of equation (5) is

$$(g_1 - g_2)^2 (g_1 - g_3)^2 \dots (g_1 - g_n)^2 (g_2 - g_3)^2 \dots (g_2 - g_n)^2 \dots \\ (g_{n-1} - g_n)^2 = 0$$

and represents a locus of double points of curves represented by (5); it is, in fact, the locus of points where a curve of type $g_r = c$ crosses a curve of type $g_s = c$. If a curve of type $g_r = c$ has double points for all values of c , the locus of these double points is apparently not given by the c -discriminant, when it is written in the above form.

EXAMPLES.

1. The differential equation

$$\left(\frac{dy}{dx}\right)^2 = \sin^2 \frac{1}{x}$$

is satisfied by

$$y = \int_0^x \sin \frac{1}{t} dt + c_1$$

$$y = - \int_0^x \sin \frac{1}{t} dt + c_2$$

and

$$y = \int_0^x \left| \sin \frac{1}{t} \right| dt + c_3$$

where c_1, c_2, c_3 are arbitrary constants. (O. Perron.)

2. Solve the equations:

$$p^2 y^2 - x^2 = 0 \quad p^2 x^2 - 3pxy + 2y^2 = 0 \quad p^2 + p(x+y) + xy = 0$$

§ 41. *Orthogonal trajectories.* Let $\varphi(x, y, p) = 0$ be the differential equation of a family of curves $f(x, y, c) = 0$, which is such that n curves pass through each point (x, y) . We shall suppose that φ is a polynomial of the n^{th} degree in p , and that there is usually one value of p for each curve through (x, y) .

Let us now consider the problem of finding a second family of curves $g(x, y, a) = 0$ defined by a differential equation $\psi(x, y, p) = 0$ of the n^{th} degree in p , and such that a (1, 1) correspondence can be set up between the directions of the tangents to the curves of the two families through each point (x, y) . We shall suppose, in fact, that the correspondence is defined by a relation of type

$$App' + Bp + Cp' + D = 0$$

where p, p' specify the directions of the tangents to the two curves and A, B, C, D are given functions of x and y . The case of chief interest occurs when $A = D, B = C = 0$, for then two corresponding tangents at P are at right angles. Since $pp' + 1 = 0$, the dif-

ferential equation $\psi(x, y, p) = 0$ is derived from $\phi(x, y, p) = 0$ by writing $-\frac{1}{p}$ in place of p . The curves defined by the equation $\psi = 0$ are then said to be the *orthogonal trajectories* of the curves defined by the equation $\phi = 0$ and *vice versa*. Thus, if

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0$$

is the differential equation of a family of curves

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0$$

is the differential equation of the orthogonal trajectories.

EXAMPLES.

1. To find the orthogonal trajectories of the family of lines

$$y = cx + b\sqrt{1 + c^2}$$

we first form the differential equation

$$y = px + b\sqrt{1 + p^2}$$

The differential equation of the orthogonal trajectories is obtained by writing $-\frac{1}{p}$ in place of p , and is

$$py + x = b\sqrt{1 + p^2}$$

To solve this equation, we write $\sqrt{1 + p^2} = \frac{ds}{dx}$, $x^2 + y^2 = r^2$; the equation then becomes

$$r \frac{dr}{dx} = b \frac{ds}{dx}$$

Transforming to polar coordinates (r, θ) , we have

$$\frac{ds}{dr} = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}$$

$$\therefore b \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} = r$$

or

$$\frac{d\theta}{dr} = \sqrt{\frac{1}{b^2} - \frac{1}{r^2}}$$

The variables are now separated, and the solution is given by

$$\theta = a + \int \frac{dr}{br} \sqrt{r^2 - b^2}$$

In this example the first system of curves consists of the tangents to the circle $x^2 + y^2 = b^2$, while the orthogonal trajectories are involutes of this circle. It should be noticed that in this case the curves of the first family have a real envelope, which is the locus of points at which the curves of the second family come to a full stop.

2. If the differential equation of a family of curves in polar coordinates is

$$f\left(\frac{d\theta}{dr}, r, \theta\right) = 0$$

the differential equation of the orthogonal trajectories is

$$f\left(-\frac{1}{r^2} \frac{dr}{d\theta}, r, \theta\right) = 0$$

3. If $u + iv = F(x + iy)$, where $i = \sqrt{-1}$, prove that the curves $v = \text{const.}$ are the orthogonal trajectories of the curves $u = \text{const.}$

4. Find the differential equation satisfied by the family of confocal conics

$$\frac{x^2}{c} + \frac{y^2}{c - \lambda} = 1$$

where λ is a constant, and show that the orthogonal trajectories also belong to this family.

5. If $r^n = x^n + y^n$, prove that the curves $\frac{\partial^n}{\partial x^n} \left(\frac{1}{r}\right) = c$ are cut orthogonally by the curves $\frac{\partial^{n+1}}{\partial x^{n+1}} = a$. (Stefan.)

6. Find the orthogonal trajectories of a system of coaxial parabolas of latus rectum $4a$.

7. Find the orthogonal trajectories of a system of similar coaxial ellipses.

§ 42. *Orthogonal trajectories of a family of circles.** Let the family of circles be given by the equation

$$[x - \xi(c)]^2 + [y - \eta(c)]^2 = [\rho(c)]^2. \quad (1)$$

where c is the variable parameter. Since the tangent at a point of an orthogonal trajectory passes through the centre of the corresponding circle through this point, the differential equation of the orthogonal trajectories is

$$\frac{dx}{x - \xi(c)} = \frac{dy}{y - \eta(c)} \quad (2)$$

where c is given by (1). Let us now write

$$x = \xi(c) + \rho(c) \frac{1 - t^2}{1 + t^2} \quad y = \eta(c) + \rho(c) \frac{2t}{1 + t^2} \quad (3)$$

and take c and t as new variables. Differentiating, we find that

$$\begin{aligned} \frac{dx}{dc} &= \xi' + \rho' \frac{1 - t^2}{1 + t^2} - \rho \frac{dt}{dc} \frac{4t}{(1 + t^2)^2} \\ \frac{dy}{dc} &= \eta' + \rho' \frac{2t}{1 + t^2} + \rho \frac{dt}{dc} \frac{2(1 - t^2)}{(1 + t^2)^2} \end{aligned}$$

* See Darboux, *Théorie des Surfaces*, t. 1, p. 113; L. Ballif, *L'Enseignement mathématique* (1915), p. 215; C. Cailler, *Ibid.* p. 223.

Multiply the first of these equations by $y - \eta$, the second by $x - \xi$, and subtract; then, after making use of (2) and (3), we get

$$\rho(c) \frac{dt}{dc} = \xi'(c)t - \frac{1}{2}\eta'(c)[1 - t^2]$$

This is a particular case of Riccati's equation; consequently we may conclude that the cross ratio of the four values of t for four orthogonal trajectories is independent of c . The geometrical meaning of t is that it is the tangent of the angle θ which the line joining the two points

$$(x, y) \quad (\xi - \rho, \eta)$$

makes with the axis of x . When c is given, these points are both on the circle (1), and the latter is independent of t ; hence the cross ratio of four values of t is equal to the cross ratio of the four corresponding points on the circle, by which we mean the cross ratio of the four lines joining these points to an arbitrary point on the circumference. Hence we have the following theorem: *Each circle of the system (1) is cut in the same cross ratio by four orthogonal trajectories.*

EXAMPLES.

1. Prove that the orthogonal trajectories of the family of circles $x^2 + y^2 - 2cx = b^2$, where b is a constant, are given by the equations

$$x = \frac{t(1 - a^2b^2)}{a(1 + t^2)} \quad y = \frac{t^2 + a^2b^2}{a(1 + t^2)}$$

or by the equation $x^2 + y^2 - 2ky + b^2 = 0$, where $2k = \frac{1 + a^2b^2}{a}$.

2. Prove that the orthogonal trajectories of the family of spheres

$$[x - \xi(c)]^2 + [y - \eta(c)]^2 + [z - \zeta(c)]^2 = [\rho(c)]^2$$

are obtained by writing

$$x = \rho(c) \frac{u + v}{1 + uv} + \xi(c) \quad y = i\rho(c) \frac{v - u}{1 + uv} + \eta(c) \quad z = \rho(c) \frac{uv - 1}{1 + uv} + \zeta(c)$$

where $i = \sqrt{-1}$, and u and v satisfy the Riccatian equations

$$\frac{du}{dc} = \frac{u^2}{2\rho}(\xi' - i\eta') - \frac{u}{\rho}\zeta' - \frac{1}{2\rho}(\xi' + i\eta')$$

$$\frac{dv}{dc} = \frac{v^2}{2\rho}(\xi' + i\eta') - \frac{v}{\rho}\zeta' - \frac{1}{2\rho}(\xi' - i\eta')$$

respectively. (Darboux.)

3. Find the orthogonal trajectories of the family of circles which touch two given straight lines.

4. Prove that when the orthogonal trajectories are known for the

system of circles (1), they can also be found for a system in which $\xi(c)$, $\eta(c)$ and $\rho(c)$ are replaced by $\xi_0(c)$, $\eta_0(c)$, $\rho_0(c)$, where

$$\xi_0'(c) = f(c)\xi'(c) \quad \eta_0'(c) = f(c)\eta'(c) \quad \rho_0(c) = f(c)\rho(c)$$

$f(c)$ being an arbitrary function. (V. Rouquet.)

5. Prove that the problem of finding the orthogonal trajectories of a family of circles can be reduced to that of finding the orthogonal trajectories of a family of equal circles. (L. Ballif.)

6. Find a curve Γ on which it is necessary to roll a line L in order that a point of a plane rigidly attached to L may describe a given curve C . Prove that Γ may be regarded as the evolute of an orthogonal trajectory of a family of equal circles whose centres lie on C . (L. Ballif.)

CHAPTER V

DIFFERENTIAL EQUATIONS WITH PARTICULAR SOLUTIONS OF A SPECIFIED TYPE

§ 43. *Euler's equation.* Since there is no general method of solving differential equations which is always applicable, and since there are many types of differential equations whose integrals cannot be expressed in a simple manner in terms of known functions, it is useful to have a list of equations whose solutions can be expressed in a simple form. The natural method of forming such a list is to start with a given type of primitive and find a corresponding differential equation.

If we start with a bilinear relation between x and y ,

$$axy + bx + cy + d = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a, b, c, d are constants, we obtain

$$(ax + c) \frac{dy}{dx} + ay + b = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and so a particular differential equation corresponding to the given primitive is

$$\frac{dx}{ax + c} + \frac{dy}{ay + b} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

an equation in which the variables are separated. The complete solution is given by (1), wherein d is the variable parameter.

If a, b, c, d are linear functions of a parameter λ , so that

$$a = a_0 + a_1\lambda \quad b = b_0 + b_1\lambda \quad c = c_0 + c_1\lambda \quad d = d_0 + d_1\lambda$$

the differential equation obtained by eliminating λ from (1) and (2) is

$$\left| \begin{array}{cc} (a_0x + c_0) \frac{dy}{dx} + a_0y + b_0 & (a_1x + c_1) \frac{dy}{dx} + a_1y + b_1 \\ a_0xy + b_0x + c_0y + d_0 & a_1xy + b_1x + c_1y + d_1 \end{array} \right| = 0$$

and is of the form $\frac{dx}{F(x)} + \frac{dy}{G(y)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$

where $F(x) = (a_0x + c_0)(b_1x + d_1) - (a_1x + c_1)(b_0x + d_0)$

$$G(y) = (a_0y + b_0)(c_1y + d_1) - (a_1y + b_1)(c_0y + d_0)$$

The general solution of (4) is furnished by (1), wherein λ is the variable parameter. It is possible, moreover, to choose the constants $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ so that $F(x)$ and $G(y)$ are arbitrarily assigned quadratic functions with equal Hessians. Notice that $F(x) = G(y)$ if $b = c$.

Next let us consider a biquadratic relation between x and y .

$$X_0 y^2 + 2X_1 y + X_2 = 0 \quad . \quad . \quad . \quad (5)$$

in which

$$X_0 = a_0 x^2 + 2a_1 x + a_2$$

$$X_1 = b_0 x^2 + 2b_1 x + b_2$$

$$X_2 = c_0 x^2 + 2c_1 x + c_2$$

and $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are constants depending on a parameter λ .

We find that

$$(X_0 y + X_1) \frac{dy}{dx} + (a_0 x + a_1) y^2 + 2(b_0 x + b_1) y + c_0 x + c_1 = 0$$

$$\text{that is} \quad (X_0 y + X_1) \frac{dy}{dx} + Y_0 x + Y_1 = 0 \quad . \quad . \quad . \quad (6)$$

where

$$Y_0 = a_0 y^2 + 2b_0 y + c_0$$

$$Y_1 = a_1 y^2 + 2b_1 y + c_1$$

$$Y_2 = a_2 y^2 + 2b_2 y + c_2$$

Squaring (6) and noticing that (5) can also be written in the form

$$Y_0 x^2 + 2Y_1 x + Y_2 = 0$$

we obtain the relation

$$(X_1^2 - X_0 X_2) \left(\frac{dy}{dx} \right)^2 = Y_1^2 - Y_0 Y_2$$

Hence, if

$$F(x) = X_1^2 - X_0 X_2 \quad G(y) = Y_1^2 - Y_0 Y_2$$

the relation (5) leads to the differential equation

$$\frac{dx}{\sqrt{F(x)}} + \frac{dy}{\sqrt{G(y)}} = 0 \quad . \quad . \quad . \quad (7)$$

It should be noticed that if $b_0 = a_1, c_0 = a_2, c_1 = b_2, F(x) = G(x)$, and the differential equation is symmetrical in x and y .

In order that the relation (5) may represent a general solution of equation (7), the differential equation must be independent of λ after it has been multiplied throughout by an appropriate function of λ . We must, therefore, make the constants a_0, b_1 , etc., depend on λ in such a way that

$$X_1^2 - X_0 X_2 = f(x) \varphi(\lambda) \quad Y_1^2 - Y_0 Y_2 = g(y) \varphi(\lambda)$$

where f and g are independent of λ . If the relation between x and y is a symmetrical one, it is only necessary to satisfy one of these

equations, and then the other is a natural consequence. Let us assume that

$$f(x) = px^4 + qx^3 + rx^2 + sx + t$$

then we have to satisfy the equations

$$\begin{aligned} b_0^2 - a_0c_0 &= p\varphi & 4b_1b_0 - 2(a_1c_0 + a_0c_1) &= q\varphi & b_2^2 - a_2c_2 &= t\varphi \\ 4b_1b_2 - 2(a_1c_2 + a_2c_1) &= s\varphi \\ 4(b_1^2 - a_1c_1) - (a_2c_0 + a_0c_2 - 2b_0b_2) &= r\varphi \end{aligned}$$

These equations are, however, difficult to solve directly, and so it is expedient to adopt another method.

Let us first take the symmetrical case and write $f(x) = X$, $f(y) = Y$. The differential equation

$$\frac{dx}{\sqrt{f(x)}} + \frac{dy}{\sqrt{f(y)}} = 0 \quad . \quad . \quad . \quad (8)$$

can be integrated by the following brief but rather artificial method.

Let us imagine x and y expressed as functions of a variable t , which is chosen in such a way that

$$\frac{dx}{dt} = \frac{X^{\frac{1}{2}}}{y - x} \quad \frac{dy}{dt} = \frac{Y^{\frac{1}{2}}}{x - y} \quad . \quad . \quad . \quad (9)$$

Equation (8) is then satisfied in virtue of equations (9). Now write

$$w = x + y$$

then

$$\frac{dw}{dt} = \frac{Y^{\frac{1}{2}} - X^{\frac{1}{2}}}{x - y}$$

$$\begin{aligned} \frac{d^2w}{dt^2} &= \frac{1}{x - y} \left\{ \frac{1}{2Y^{\frac{1}{2}}} \frac{dY}{dy} \frac{dy}{dt} - \frac{1}{2X^{\frac{1}{2}}} \frac{dX}{dx} \frac{dx}{dt} \right\} - \frac{Y^{\frac{1}{2}} - X^{\frac{1}{2}}}{(x - y)^2} \left(\frac{dx}{dt} - \frac{dy}{dt} \right) \\ &= \frac{1}{(x - y)^2} \left\{ \frac{1}{2} \frac{dY}{dy} + \frac{1}{2} \frac{dX}{dx} - \frac{Y - X}{y - x} \right\} \\ &= \frac{1}{(x - y)^2} \{ \frac{1}{2}q(x^2 - 2xy + y^2) + p(x^3 - x^2y - xy^2 + y^3) \} \\ &= \frac{1}{2}q + p(x + y) = \frac{1}{2}q + pw \end{aligned}$$

Multiplying the equation

$$\frac{d^2w}{dt^2} = pw + \frac{1}{2}q$$

by the integrating factor $2 \frac{dw}{dt}$ and integrating, we obtain

$$\left(\frac{dw}{dt} \right)^2 = pw^2 + qw + \lambda$$

where λ is an arbitrary constant. Hence a solution of (8) is given by

$$\frac{(Y^{\frac{1}{2}} - X^{\frac{1}{2}})^2}{(y - x)^2} = p(x + y)^2 + q(x + y) + \lambda \quad . \quad . \quad (10)$$

The unsymmetrical equation

$$\frac{dx}{\sqrt{f(x)}} + \frac{dy}{\sqrt{g(y)}} = 0 \quad . \quad . \quad . \quad . \quad (11)$$

is best treated by a geometrical method.

Let x and y be regarded as circular coordinates, that is, coordinates connected with the usual Cartesian coordinates x_0 and y_0 by the $x = x_0 + iy_0$, $y = x_0 - iy_0$. Equation (5) then represents a bicircular quartic curve, that is, a quartic curve having double points at the circular points at infinity. The condition $X_0X_2 - X_1^2 = 0$ then states that for a value of x satisfying this equation the two values of y given by (5) coincide. Hence $X_0X_2 - X_1^2 = 0$ is the equation of the four tangents from one circular point to the curve, and, similarly, $Y_0Y_2 - Y_1^2 = 0$ is the equation of the four tangents from the other circular point. These two sets of four tangents intersect in the sixteen foci of the curve; hence, if we are given $f(x)$ and $g(y)$, we are given the foci of the bicircular quartic, and the problem of finding all the relations of type (5), which are equivalent to a given differential equation (11), is the same as that of finding all the bicircular quartics with given foci.

It should be remarked at the outset that a differential equation of type (11) only possesses solutions of the form (5) when certain conditions are satisfied. In the first place, the cross ratio of the four tangents from one node to a binodal quartic is known to be equal to the cross ratio of the four tangents from the other node; hence the cross ratio of the four roots of $f(x)$ must be equal to the cross ratio of the four roots of $g(y)$. Secondly, we can generally make the relation between x and y symmetrical by transforming (5) by a substitution of type

$$y = \frac{\alpha y_1 + \beta}{\gamma y_1 + \delta} \quad . \quad . \quad . \quad . \quad (12)$$

If U_0, U_1, U_2 are the coefficients of $y_1^2, 2y_1, 1$ in the new equation, it is easy to see that

$$U_0 = \alpha^2 X_0 + 2\alpha\gamma X_1 + \gamma^2 X_2$$

$$U_1 = \alpha\beta X_0 + (\alpha\delta + \beta\gamma)X_1 + \gamma\delta X_2$$

$$U_2 = \beta^2 X_0 + 2\beta\delta X_1 + \delta^2 X_2$$

and
$$U_0U_2 - U_1^2 = (\alpha\delta - \beta\gamma)^2(X_0X_2 - X_1^2)$$

Hence, if (12) is the substitution which makes (5) symmetrical, we have

$$(\gamma y_1 + \delta)^4 g\left(\frac{\alpha y_1 + \beta}{\gamma y_1 + \delta}\right) = (\alpha\delta - \beta\gamma)^2 f(y_1) \quad . \quad . \quad (13)$$

This condition includes the other one; hence, in order that the differential equation (11) should admit a solution of type (5), it is necessary that the functions $f(x)$ and $g(y)$ should satisfy a relation

of the above type.* The condition is also sufficient, for we have

$$dy = \frac{\alpha\delta - \beta\gamma}{(\gamma y_1 + \delta)^2} dy_1$$

hence the differential equation (11) is transformed into

$$\frac{dx}{\sqrt{f(x)}} + \frac{dy_1}{\sqrt{f(y_1)}} = 0$$

and so a solution of type (10) with y_1 instead of y can be obtained by the preceding method. It remains to be shown that equation (10) is equivalent to an equation of the second degree in both x and y . The rationalised form of (10) is

$$(y - x)^4[\lambda + q(x + y) + p(x + y)^2] - 2(X + Y)(y - x)^2 \\ \times \{p(x + y)^2 + q(x + y) + \lambda\} + (Y - X)^2 = 0$$

Substituting the values of X and Y and rejecting the factor $(x - y)^2$, we find after a little reduction that

$$\left. \begin{aligned} &\lambda^2(x - y)^2 \\ &- 2\lambda[2px^2y^2 + qxy(x + y) + r(x^2 + y^2) + s(x + y) + 2t] \\ &+ q^2x^2y^2 + r^2(x + y)^2 + 2qrx y(x + y) \\ &+ s^2 - 2qsxy + 2rs(x + y) - 4qt(x + y) \\ &- 4psxy(x + y) - 4pt(x + y)^2 = 0 \end{aligned} \right\} \quad (14)$$

A solution of the differential equation (11) may also be obtained by the following geometrical method.

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$ be the roots of the equations $f(x) = 0$, $g(y) = 0$ respectively, and let them be arranged so that the cross ratios, which are known to be equal, are

$$\frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_4)} = \frac{(\beta_2 - \beta_3)(\beta_1 - \beta_4)}{(\beta_3 - \beta_1)(\beta_2 - \beta_4)}$$

The four foci whose circular coordinates are (α_1, β_1) , (α_2, β_2) , (α_3, β_3) , (α_4, β_4) then lie on a circle.† Let the equation of this circle be

$$xy + ux + vy + k = 0 \quad \dots \quad (15)$$

Now a bicircular quartic is the envelope of a circle which is orthogonal to a given circle C and whose centre lies on a fixed conic Γ : the points of intersection of Γ and C are, moreover, foci of the bicircular quartic. Hence, if we wish to obtain the general bicircular quartic with the four given foci, we must take the conic Γ to be the general conic through the four given points. Let the equation of this conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy \\ + c + 2\lambda(xy + ux + vy + k) = 0 \quad \dots \quad (16)$$

*The condition is satisfied when the invariants I and J of the functions $f(x)$ and $g(y)$ are equal.

†In an exceptional case they may lie on a straight line; the present method then requires modification.

If (x_0, y_0) are the circular coordinates of an arbitrary point on this conic, the equation of a circle with this point as centre is

$$xy - y_0x - x_0y + \rho = 0 \quad (17)$$

This circle cuts (15) orthogonally if

$$ux_0 + vy_0 - k - \rho = 0$$

hence equation (17) can be written in the form

$$x_0(u + y) + y_0(v + x) + k - xy = 0 \quad (18)$$

We have now to find the envelope of this circle subject to the condition

$$ax_0^2 + 2(h + \lambda)x_0y_0 + by_0^2 + 2(g + \lambda u)x_0 + 2(f + \lambda v)y_0 + c + 2\lambda k = 0 \quad (19)$$

To do this we must eliminate y_0 and find the condition that the resulting equation in x_0 has equal roots. This process is exactly that used in finding the condition that a line should touch a conic when (x_0, y_0) are current coordinates; hence the equation of the envelope of the circle (18) is

$$\begin{aligned} & [b(c + 2\lambda k) - (f + \lambda v)^2](u + y)^2 + [a(c + 2\lambda k) - (g + \lambda u)^2](v + x)^2 \\ & + [ab - (h + \lambda)^2](k - xy)^2 \\ & + 2[(g + \lambda u)(h + \lambda) - a(f + \lambda v)](v + x)(k - xy) \\ & + 2[(f + \lambda v)(h + \lambda) - b(g + \lambda u)](u + y)(k - xy) \\ & + 2[(f + \lambda v)(g + \lambda u) - (c + 2\lambda k)(h + \lambda)](u + y)(v + x) = 0 \quad (20) \end{aligned}$$

and this is a general solution of the differential equation, λ being an arbitrary parameter. In the preceding analysis the constants $a, b, c, f, g, h, u, v, k$ are supposed to be known in terms of the roots of $f(x), g(y)$, but we can obtain a differential equation from the relation (20), and then identify it with the given equation.

Let us first of all put $\lambda = 0$ in (20), then the resulting equation can be written in the two forms

$$\begin{aligned} y^2X_0 + 2yX_1 + X_2 &= 0 \\ x^2Y_0 + 2xY_1 + Y_2 &= 0 \end{aligned}$$

where

$$X_0 = A - 2Gx + Cx^2$$

$$X_1 = Au - kCx - Fx(v + x) - G(ux - k) + H(v + x)$$

$$X_2 = Au^2 + B(v + x)^2 + Ck^2 + 2kF(v + x) + 2kGu + 2Hu(v + x)$$

and A, B, C, F, G, H are the minors of a, b, c, f, g, h in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The expressions for Y_0, Y_1, Y_2 are obtained by writing y, B, F, v for x, A, G, u respectively and *vice versa*.

Since $BC - F^2 = a\Delta$, $GH - AF = f\Delta$, etc., it follows that

$$\begin{aligned} X_0X_2 - X_1^2 &= \Delta[ax^2(v+x)^2 + b(ux+k)^2 + c(v+x)^2 \\ &\quad - 2f(v+x)(ux+k) + 2gx(v+x)^2 \\ &\quad - 2hx(v+x)(ux+k)] = \Delta f(x) \quad (21) \end{aligned}$$

$$\begin{aligned} Y_0Y_2 - Y_1^2 &= \Delta[avy + k)^2 + by^2(u+y)^2 + c(u+y)^2 \\ &\quad + 2fy(u+y)^2 - 2g(u+y)(vy+k) \\ &\quad - 2hy(u+y)(vy+k)] = \Delta g(y) \quad (22) \end{aligned}$$

and so the differential equation is

$$\frac{dx}{\sqrt{f(x)}} \pm \frac{dy}{\sqrt{g(y)}} = 0 \quad (23)$$

When $\lambda \neq 0$ we must write $a, b, h + \lambda, f + \lambda v, g + \lambda u, c + 2\lambda k$ for a, b, h, f, g, c respectively, but the differential equation is unaltered for

$$\begin{aligned} 2\lambda k(v+x)^2 - 2\lambda v(v+x)(ux+k) \\ + 2\lambda u(v+x)^2x - 2\lambda x(v+x)(ux+k) \equiv 0 \end{aligned}$$

Hence, if $f(x)$ and $g(y)$ are defined by (21) and (22), the solution of the differential equation (23) is given by (20). If we arrange (20) according to powers of λ , it takes the form

$$\begin{aligned} (bc - f^2)(u+y)^2 + (ac - g^2)(v+x)^2 + (ab - h^2)(k - xy)^2 \\ + 2(gh - af)(v+x)(k - xy) + 2(hf - bg)(u+y)(k - xy) \\ + 2(fg - ch)(u+y)(v+x) + 2\lambda[(bk - fv)(u+y)^2 \\ + (ak - gu)(v+x)^2 - h(k - xy)^2 \\ + (hu - av + g)(v+x)(k + xy) \\ + (vh - bu + f)(u+y)(k - xy) \\ + (uf + vg - c - 2hk)(u+y)(v+x)] \\ - \lambda^2(xy + ux + vy + k)^2 = 0 \end{aligned}$$

EXAMPLES.

1. Show that the relation

$$\begin{aligned} (x^3 - y^3)^2 - 2\lambda^2[x^3 + y^3 - 2(1 + k^2)x^2y^2 + k^2x^2y^2(x^3 + y^3)] \\ + \lambda^4(1 - k^2x^2y^2)^2 = 0 \end{aligned}$$

is a solution of the differential equation

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 0$$

Show also that the solution can be expressed in the form

$$x\sqrt{1-y^2}\sqrt{1-k^2y^2} + y\sqrt{1-x^2}\sqrt{1-k^2x^2} = \lambda(1 - k^2x^2y^2)$$

2. If the function $\wp(u)$ is defined by the property that $x = \wp(u)$ when

$$u = \int_x^\infty (4t^3 - g_2t - g_3)^{-\frac{1}{2}} dt \quad (g_2, g_3 \text{ constants})$$

show that
$$\wp(u+v) + \wp(u) + \wp(v) = \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2$$

3. If $\varphi(x)$, $\psi(x)$ are polynomials of the second degree, the equation

$$\frac{dx}{\sqrt{\varphi(x)\psi(x)}} = \frac{dy}{\sqrt{\varphi(y)\psi(y)}}$$

is satisfied by
$$\frac{\sqrt{\varphi(x)\psi(y)} - \sqrt{\varphi(y)\psi(x)}}{x - y} = \text{const.} \quad (\text{Laguerre.})$$

4. The equation

$$\frac{dx}{\sqrt{x^4 + 2ax^2 + 1}} \pm \frac{dy}{\sqrt{y^4 + 2ay^2 + 1}} = 0$$

is satisfied by
$$\frac{(xy + 1)^2}{1 + \lambda} + \frac{(xy - 1)^2}{1 - \lambda} + \frac{(x + y)^2}{a - \lambda} = 0$$

Prove that if x and y are regarded as the parameters of the isotropic generators of the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$, so that

$$\xi = i \frac{xy + 1}{x - y} \quad \eta = \frac{xy - 1}{x - y} \quad \zeta = \frac{x + y}{x - y}$$

the integral curves of the differential equation are confocal sphero-conics. (E. Turrière.)

5. Prove that equation (23) can be made symmetrical in x and y , by the substitution

$$yy_1 + uy_1 + vy + k = 0$$

Notice that this is determined by the circle (15).

6. When Euler's differential equation can be reduced to the symmetrical form this can generally be done in four different ways.

7. Prove that when Euler's differential equation is rationalised the p -discriminant and the λ -discriminant both give the singular solution $f(x)g(y) = 0$, which represents the envelope of the family of confocal bicircular quartics.

§ 44. *Linear differential equations with simple solutions.* No method of expressing the solution of the general linear differential equation of the second order in finite terms has yet been discovered. We shall therefore give a brief outline of the properties of some well-known equations which possess particular solutions of a simple type, for we have seen that when a particular solution is known the general solution can be found by integration (§ 30).

If a , b , λ , μ are constants, the function $y = (x - a)^\lambda (x - b)^\mu$ satisfies a number of linear differential equations of the second order of which

$$\begin{aligned} (x - a)(x - b) \frac{d^2 y}{dx^2} - [(\lambda - 1)(x - b) + (\mu - 1)(x - a)] \frac{dy}{dx} \\ - (\lambda + \mu)y = 0 \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

is the most interesting. Differentiating this equation ν times with regard to x and using Leibnitz's theorem, we obtain

$$(x-a)(x-b)\frac{d^2u}{dx^2} - [(\lambda - \nu - 1)(x-b) + (\mu - \nu - 1)(x-a)]\frac{du}{dx} + [\nu^2 - \nu(\lambda + \mu - 1) - \lambda - \mu]u = 0 \quad (2)$$

where
$$u = \frac{d^\nu y}{dx^\nu} = \frac{d^\nu}{dx^\nu} (x-a)^\lambda (x-b)^\mu$$

In particular, if $a = 0$, $b = 1$, we find that the equation

$$x(1-x)\frac{d^2u}{dx^2} + [\nu + 1 - \lambda - (2\nu + 2 - \lambda - \mu)x]\frac{du}{dx} - (\nu + 1)(\nu - \lambda - \mu)u = 0 \quad (3)$$

is satisfied by the function

$$u = \frac{d^\nu}{dx^\nu} x^\lambda (1-x)^\mu$$

The differential equation

$$x(1-x)\frac{d^2u}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{du}{dx} - \alpha\beta u = 0 \quad (4)$$

is called the *hypergeometric equation*, it becomes identical with (3)

when $\alpha = \nu + 1$ $\beta = \nu - \lambda - \mu$ $\gamma = \nu + 1 - \lambda$

We may conclude then that, if α is a *positive integer*, the hypergeometric equation is satisfied by the function

$$u = \frac{d^{\alpha-1}}{dx^{\alpha-1}} x^{\alpha-\gamma} (1-x)^{\gamma-\beta-1} \quad (5)$$

If we put $u = x^{1-\gamma}v$, equation (4) becomes

$$x(1-x)\frac{d^2v}{dx^2} + [2 - \gamma - (\alpha + \beta - 2\gamma + 3)x]\frac{dv}{dx} - (\alpha - \gamma + 1)(\beta - \gamma + 1)v = 0$$

This is also a hypergeometric equation, and is consequently satisfied by

$$v = \frac{d^{\alpha-\gamma}}{dx^{\alpha-\gamma}} x^{\alpha-1} (1-x)^{-\beta}$$

when $\alpha - \gamma + 1$ is a *positive integer*. Hence it follows that under the same condition equation (4) is satisfied by

$$u = x^{1-\gamma} \frac{d^{\alpha-\gamma}}{dx^{\alpha-\gamma}} x^{\alpha-1} (1-x)^{-\beta} \quad (6)$$

If $|x| < 1$, we can expand $(1-x)^{-\beta}$ by the binomial theorem and differentiate the product term by term; if then γ is not a negative integer, we obtain

$$u = (\alpha - 1)(\alpha - 2) \dots \gamma \left[1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)} x^2 + \dots \right]$$

The series within square brackets is called the *hypergeometric function* and is usually denoted by the symbol $F(\alpha, \beta, \gamma, x)$. It will be shown in Chapter IX that the series representing $F(\alpha, \beta, \gamma, x)$ satisfies equation (4) for unrestricted values of α, β, γ .

When the expression (5) is evaluated in a similar way, we find that

$$\begin{aligned} u &= \frac{d^{\alpha-1}}{dx^{\alpha-1}} x^{\alpha-\gamma} (1-x)^{\gamma-\beta-1} \\ &= (\alpha-\gamma)(\alpha-\gamma-1) \dots (2-\gamma) \cdot x^{1-\gamma} \\ &\quad \times F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x) \end{aligned}$$

hence $x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$

is a solution of the hypergeometric equation. Similarly, by making the substitution $u = (1-x)^{\gamma-\alpha-\beta} v$, we can prove that when $\gamma-\alpha$ is a positive integer

$$u = (1-x)^{\gamma-\alpha-\beta} \frac{d^{\gamma-\alpha-1}}{dx^{\gamma-\alpha-1}} x^{-\alpha} (1-x)^{\beta-1} \dots \quad (7)$$

is a solution of (4), and that when $1-\alpha$ is a positive integer

$$u = x^{1-\gamma} (1-x)^{\gamma-\alpha-\beta} \frac{d^{-\alpha}}{dx^{-\alpha}} x^{\gamma-\alpha-1} (1-x)^{\beta-\gamma} \dots \quad (8)$$

is a solution. Hence it follows that in these cases

$$x^{1-\gamma} (1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma, x)$$

and

$$(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$

are solutions of the hypergeometric equation (4).

It is clear that α and β can be interchanged in any of the above results; hence we may conclude that a solution of equation (4) can be expressed by means of a formula analogous to one of those already given if any one of the quantities $\alpha, \beta, 1-\alpha, 1-\beta, \gamma-\alpha, \gamma-\beta, \alpha-\gamma+1, \beta-\gamma+1$ is a positive integer.

If in equation (4) we put * $x = \frac{s}{\beta}$ and make $\beta \rightarrow \infty$, we obtain the equation

$$s \frac{d^2 u}{ds^2} + (\gamma - s) \frac{du}{ds} - \alpha u = 0 \dots \quad (9)$$

which has already been chosen as the canonical form of an equation of Laplace's type. Since

$$\lim_{\beta \rightarrow \infty} \left(1 - \frac{s}{\beta}\right)^{-\beta} = e^s$$

we may make the following deductions from the preceding results.

* This device was used by Kummer in his well-known paper on the hypergeometric series, *Crelle's Journal*, Bd. 15 (1836), pp. 39-83, 127-172. See especially p. 138. It will be unnecessary to make a rigorous examination of the limiting processes, as the results can easily be verified.

1°. If α is a positive integer, a solution of (9) is given by

$$u = \frac{d^{\alpha-1}}{ds^{\alpha-1}} [s^{\alpha-\gamma} e^s] \quad \dots \quad (10)$$

2°. If $\alpha - \gamma + 1$ is a positive integer, a solution of (9) is given by

$$u = s^{1-\gamma} \frac{d^{\alpha-\gamma}}{ds^{\alpha-\gamma}} [s^{\alpha-1} e^s] \quad \dots \quad (11)$$

3°. If $\gamma - \alpha$ is a positive integer, a solution of (9) is given by

$$u = e^s \frac{d^{\gamma-\alpha-1}}{ds^{\gamma-\alpha-1}} [s^{-\alpha} e^{-s}] \quad \dots \quad (12)$$

4°. If $1 - \alpha$ is a positive integer, a solution of (9) is given by

$$u = s^{1-\gamma} e^s \frac{d^{-\alpha}}{ds^{-\alpha}} [s^{\gamma-\alpha-1} e^{-s}] \quad \dots \quad (13)$$

The limiting form of the series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} \frac{s}{\beta} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2\gamma(\gamma+1)} \left(\frac{s}{\beta}\right)^2 + \dots$$

is
$$F(\alpha, \gamma, s) = 1 + \frac{\alpha}{1 \cdot \gamma} s + \frac{\alpha(\alpha+1)}{1 \cdot 2\gamma(\gamma+1)} s^2 + \dots \quad (14)$$

It will be shown in Chapter IX. that this series satisfies equation (9) for unrestricted values of α and γ . It should be noticed that if $1 - \alpha$ is a positive integer the series terminates; it may also be deduced from 4° that in this case the equation possesses a solution which is a polynomial of degree $-\alpha$ in s .

If in (2) we write $a = 1$, $b = -1$, $\lambda = \mu = \nu = n$, we obtain Legendre's equation

$$(1 - x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + n(n+1)u = 0 \quad \dots \quad (15)$$

which is satisfied by the function

$$u = \frac{d^n}{dx^n} [(x^2 - 1)^n] \quad \dots \quad (16)$$

when n is a positive integer. This function is usually denoted by $2^n n! P_n(x)$, and is a polynomial of the n^{th} degree.

Let us next consider Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \dots \quad (17)$$

which is reduced to the form

$$x \frac{d^2 u}{dx^2} + (2n+1) \frac{du}{dx} + xu = 0 \quad \dots \quad (18)$$

by the substitution $y = x^n u$. To reduce this to Weiler's canonical form for an equation of Laplace's type, we write

$$u = v e^{ix} \quad x = \frac{1}{2} i \xi \quad \text{where } i = \sqrt{-1}$$

The new equation is

$$\xi \frac{d^2 v}{d\xi^2} + (2n + 1 - \xi) \frac{dv}{d\xi} - (n + \frac{1}{2})v = 0 \quad (19)$$

and is of the type (9) if $\alpha = n + \frac{1}{2}$, $\gamma = 2n + 1$. It follows then from 1° and 3° that if $n + \frac{1}{2}$ is a positive integer, the differential equation is satisfied by the two expressions

$$v = \frac{d^m}{d\xi^m} [\xi^{-m-1} e^\xi] \quad (20)$$

$$v = e^\xi \frac{d^m}{d\xi^m} [\xi^{-m-1} e^{-\xi}] \quad (21)$$

where $m = n - \frac{1}{2}$. We thus find that Bessel's equation is satisfied by the function

$$y = e^{ix} x^n \frac{d^m}{dx^m} (x^{-m-1} e^{-2ix}) \quad (22)$$

Let us write

$$K_{m+\frac{1}{2}}(x) = (-1)^m 2^{-m} \sqrt{\frac{\pi}{2}} e^x \cdot x^{m+\frac{1}{2}} \frac{d^m}{dx^m} (x^{-m-1} e^{-2x}) \quad (23)$$

when m is a positive integer; then, if $n = m + \frac{1}{2}$, we have the result that Bessel's equation is satisfied by the functions

$$K_{m+\frac{1}{2}}(ix) \quad K_{m+\frac{1}{2}}(-ix)$$

A solution of Bessel's equation for this case can also be obtained in the following way. If we write $x = \sqrt{s}$ in (18), the equation becomes

$$s \frac{d^2 u}{ds^2} + (n + 1) \frac{du}{ds} + \frac{1}{4} u = 0 \quad (24)$$

Differentiating this equation m times with regard to s and putting $v = \frac{d^m u}{ds^m}$, we find that v satisfies the equation

$$s \frac{d^2 v}{ds^2} + (n + m + 1) \frac{dv}{ds} + \frac{1}{4} v = 0 \quad (25)$$

which is of the same type as (24). Now put $n = -\frac{1}{2}$, then (18) becomes $\frac{d^2 u}{dx^2} + u = 0$ and is satisfied by $u = \cos x$ and $u = \sin x$; hence it follows that with this value of n equation (24) is satisfied by $\cos \sqrt{s}$ and $\sin \sqrt{s}$, and (25) by

$$\frac{d^m}{ds^m} (\cos \sqrt{s}) \quad \text{and} \quad \frac{d^m}{ds^m} (\sin \sqrt{s})$$

This means that the last two expressions are solutions of the equation

$$s \frac{d^2 v}{ds^2} + (m + \frac{1}{2}) \frac{dv}{ds} + \frac{1}{4} v = 0$$

consequently, if $n + \frac{3}{2}$ is a positive integer, two solutions of Bessel's equation are

$$x^n \frac{d^m}{ds^m} (\cos \sqrt{s}) \quad \text{and} \quad x^n \frac{d^m}{ds^m} (\sin \sqrt{s})$$

where $m = n + \frac{1}{2}$, $s = x^2$. In the usual notation

$$J_{m+\frac{1}{2}}(x) = (-1)^{m+1} \frac{2}{\sqrt{\pi}} (2x)^{m+\frac{1}{2}} \frac{d^{m+1}}{ds^{m+1}} (\cos \sqrt{s})$$

$$Y_{m+\frac{1}{2}}(x) = (-1)^{m+1} \frac{2}{\sqrt{\pi}} (2x)^{m+\frac{1}{2}} \frac{d^{m+1}}{ds^{m+1}} (\sin \sqrt{s})$$

When $n = m + \frac{1}{2}$, the functions $J_n(x)$ and $Y_n(x)$ are independent solutions of Bessel's equation. When $n = \frac{1}{2}$, we have the two independent solutions

$$\frac{\cos x}{\sqrt{x}} \quad \text{and} \quad \frac{\sin x}{\sqrt{x}}$$

EXAMPLES.

1. Prove that $P_n(x) = F\left(n+1, -n, 1, \frac{1-x}{2}\right)$

2. If a is a constant and $p+2$ a positive integer, the equation

$$\frac{d^2 u}{dx^2} - a^2 u = \frac{p(p+1)}{x^2} u$$

is satisfied by $u = x^{p+1} \left(\frac{1}{x} \frac{d}{dx}\right)^{p+1} \left(\frac{c_1 e^{ax} + c_2 e^{-ax}}{x}\right)$. (Gaskin.)

3. Prove that the roots of the polynomial $P_n(x)$ are all real and lie in the interval $(-1, +1)$.

4. If $1-\alpha$ is a positive integer, the roots of the polynomial $F(\alpha, \gamma, s)$ are all real and positive. (Gegenbauer).

5. Prove that if $m+1$ is a positive integer,

$$J_{m+\frac{1}{2}}(x) = (-1)^m \frac{1}{\sqrt{\pi}} (2x)^{m+\frac{1}{2}} \frac{d^m}{d(x^2)^m} \left(\frac{\sin x}{x}\right)$$

$$K_{m+\frac{1}{2}}(x) = (-1)^{m+1} \sqrt{\pi} (2x)^{m+\frac{1}{2}} \frac{d^{m+1}}{ds^{m+1}} (e^{-\sqrt{s}})$$

where $s = x^2$.

6. Prove that the equation

$$(x-a)^2(x-b)^2 \frac{d^2 y}{dx^2} + cy = 0$$

is satisfied by

$$y = A(x-a)^m(x-b)^n + B(x-a)^n(x-b)^m$$

where A and B are arbitrary constants and m and n are the roots of the equation

$$\rho^2 - \rho + \frac{c}{(a-b)^2} = 0 \quad (\text{Stokes.})$$

CHAPTER VI

PARTIAL DIFFERENTIAL EQUATIONS

§ 45. *Functional determinants and their properties.* Before we can commence a treatment of partial differential equations a preliminary theorem is needed which will enable us to ascertain when n functions $u_1, u_2, \dots, u_n$ of the n independent variables $x_1, x_2, \dots, x_n$ are connected by a relation of type

$$F(u_1, u_2, \dots, u_n) = 0 \quad (1)$$

which does not involve $x_1, x_2, \dots, x_n$ explicitly. If such a relation exists, we have the n relations

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial x_1} + \dots + \frac{\partial F}{\partial u_n} \frac{\partial u_n}{\partial x_1} = 0$$

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial x_2} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial x_2} + \dots + \frac{\partial F}{\partial u_n} \frac{\partial u_n}{\partial x_2} = 0$$

$$\dots \dots \dots$$

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial x_n} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial x_n} + \dots + \frac{\partial F}{\partial u_n} \frac{\partial u_n}{\partial x_n} = 0$$

which are obtained by differentiating (1) with regard to $x_1, x_2, \dots, x_n$ respectively. If now we eliminate $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \dots, \frac{\partial F}{\partial u_n}$ from these equations, we find that

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

Let us denote the determinant on the left-hand side by the symbol J , and call it the *Jacobian* or functional determinant of the

functions $u_1, u_2, \dots u_n$ with respect to the variables $x_1, x_2, \dots x_n$. It will be convenient to use the notation

$$J = \frac{\partial(u_1, u_2, \dots u_n)}{\partial(x_1, x_2, \dots x_n)}$$

The vanishing of J is seen to be a necessary condition for the existence of a relation of type (1); we must next show that it is a sufficient condition. The following property of the functional determinant is needed for this purpose.

If J is the Jacobian of $u_1, u_2, \dots u_n$ when considered as functions of $x_1, x_2, \dots x_n$ and J_1 is the Jacobian of $x_1, x_2, \dots x_n$ when these quantities are considered as functions of the n independent variables $y_1, y_2, \dots y_n$, then JJ_1 is the Jacobian of $u_1, u_2, \dots u_n$ when these quantities are considered as functions of $y_1, y_2, \dots y_n$. In other words,

$$\frac{\partial(u_1, u_2, \dots u_n)}{\partial(x_1, x_2, \dots x_n)} \cdot \frac{\partial(x_1, x_2, \dots x_n)}{\partial(y_1, y_2, \dots y_n)} = \frac{\partial(u_1, u_2, \dots u_n)}{\partial(y_1, y_2, \dots y_n)} \quad (2)$$

This is proved by multiplying the first two determinants together by the ordinary rule and making use of the n^2 relations of type

$$\frac{\partial u_r}{\partial y_s} = \frac{\partial u_r}{\partial x_1} \frac{\partial x_1}{\partial y_s} + \frac{\partial u_r}{\partial x_2} \frac{\partial x_2}{\partial y_s} + \dots \frac{\partial u_r}{\partial x_n} \frac{\partial x_n}{\partial y_s} \quad (3)$$

In particular, we have

$$\frac{\partial(y_1, y_2, \dots y_n)}{\partial(x_1, x_2, \dots x_n)} \frac{\partial(x_1, x_2, \dots x_n)}{\partial(y_1, y_2, \dots y_n)} = 1 \quad (4)$$

Let us now suppose that the Jacobian J of the n quantities $u_1, u_2, \dots u_n$ with respect to $x_1, x_2, \dots x_n$ vanishes identically, we shall endeavour to prove that in consequence of this the quantities $u_1, u_2, \dots u_n$ are connected by a relation of type (1).

If the $n - 1$ functions $u_1, u_2, \dots u_{n-1}$ be not independent of one another, then the proposition to be proved is at once granted; we may therefore suppose them to be independent of one another.

Between the n equations expressing the n functions u we can eliminate * $n - 1$ of the variables; if the remaining variable, say x_n , be not eliminated thereby, the result may be written in the form

$$u_n = \varphi(u_1, u_2, \dots u_{n-1}, x_n) \quad (5)$$

If x_n is a function of $u_1, u_2, \dots u_{n-1}$ only, we may substitute its value in the preceding equation and obtain the desired relation between $u_1, u_2, \dots u_n$. If not, we can assume that $u_1, u_2, \dots u_{n-1}, x_n$ are independent, and we are entitled to write down the relation

$$\frac{\partial(u_1, u_2, \dots u_{n-1}, \varphi)}{\partial(u_1, u_2, \dots u_{n-1}, x_n)} \times \frac{\partial(u_1, u_2, \dots u_{n-1}, x_n)}{\partial(x_1, x_2, \dots x_{n-1}, x_n)} = J = 0$$

* Our proof is, to some extent, incomplete, because no method has been indicated by means of which this elimination can actually be carried out in the general case. To overcome this difficulty, certain existence theorems are required, and these are too difficult to be considered here. We must refer the reader to Goursat's *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Paris (1891).

Since the functions $u_1, u_2, \dots, u_{n-1}$ are independent, the first factor on the left-hand side is simply $\frac{\partial \phi}{\partial x_n}$; if this is zero, we find on integrating with regard to x_n that ϕ does not contain x_n , and consequently (5) indicates that there is a relation between $u_1, u_2, \dots, u_{n-1}$. If $\frac{\partial \phi}{\partial x_n}$ is not zero, the second factor on the left-hand side must be zero, and this gives us the equation

$$J_1 = \frac{\partial(u_1, u_2, \dots, u_{n-1})}{\partial(x_1, x_2, \dots, x_{n-1})} = 0$$

Treating this equation in the same way as $J = 0$, we find that there is either a relation between $u_1, u_2, \dots, u_{n-1}$ considered as functions of $x_1, x_2, \dots, x_{n-1}$ or the Jacobian

$$J_2 = \frac{\partial(u_1, u_2, \dots, u_{n-2})}{\partial(x_1, x_2, \dots, x_{n-2})}$$

is zero. In the first case the relation will be of the form

$$\psi(u_1, u_2, \dots, u_{n-1}, x_n) = 0 \quad \dots \quad (6)$$

but the existence of such a relation has already been excluded. In obtaining the relation $J_1 = 0$ we assume that $u_1, u_2, \dots, u_{n-2}$ are independent, and we eliminate $x_1, x_2, \dots, x_{n-2}$ between the equations defining $u_1, u_2, \dots, u_{n-1}$. Let the resulting equation be

$$u_{n-1} = \phi_1(u_1, u_2, \dots, u_{n-2}, x_{n-1}, x_n) \quad \dots \quad (7)$$

It is necessary also to assume that $u_1, u_2, \dots, u_{n-2}, x_{n-1}$ are independent in order that the first of the two Jacobians, which when multiplied together give J_1 , may have a meaning. If, however, there were a relation of type

$$x_{n-1} = \psi_1(u_1, u_2, \dots, u_{n-2}, x_n) \quad \dots \quad (8)$$

we could deduce from (7) and (8) that there was a relation of type (6), and this possibility has already been considered.

Proceeding in this way step by step we find eventually that either there is a relation between $u_1, u_2, \dots, u_n$ or $\frac{\partial u_1}{\partial x_1} = 0$. Arranging the u 's and x 's in different orders and proceeding as before, we see that the alternative to there being a relation between $u_1, u_2, \dots, u_n$ is that all the quantities of type $\frac{\partial u_r}{\partial x_s}$ should vanish. This, however, is impossible unless all the u 's are constants; hence there must be a relation between $u_1, u_2, \dots, u_n$.

Now consider a system of equations

$$u_1 = a_1, \quad u_2 = a_2, \quad \dots \quad u_n = a_n \quad \dots \quad (9)$$

where the a 's are constants. If the Jacobian of the u 's vanishes identically, we can say that the equations are inconsistent with one another unless the a 's satisfy a certain relation, and when this

relation is satisfied the equations are not independent. To prove this, we remark that since $J \equiv 0$ there is a relation of the form

$$F(u_1, u_2, \dots, u_n) = 0$$

consequently the equations (9) can only be satisfied simultaneously when the condition $F(a_1, a_2, \dots, a_n) = 0$

is satisfied. If the relation between the u 's is written in the form

$$u_n - a_n = f[u_1 - a_1, u_2 - a_2, \dots, u_{n-1} - a_{n-1}]$$

the condition is simply $f(0, 0, \dots, 0) = 0$; hence it follows that when the condition is satisfied, the relation $u_n = a_n$ is a consequence of the other relations $u_1 = a_1, u_2 = a_2, \dots, u_{n-1} = a_{n-1}$.

Note. If the equation $J = 0$ is not satisfied identically, but only in virtue of some or all of the equations $u_1 = a_1, u_2 = a_2, \dots, u_n = a_n$, we cannot infer that there is a relation between $u_1, u_2, \dots, u_n$, and so the preceding conclusions are no longer valid.

Example. If $u_1 = x_1^2 + x_2^2 - 1, u_2 = x_1 \cos \alpha + x_2 \sin \alpha - 1$, where α is a constant, we have

$$J \equiv \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} = 2(x_1 \sin \alpha - x_2 \cos \alpha)$$

In this case J is not identically zero, but it is zero when $u_1 = 0, u_2 = 0$, for

$$\begin{aligned} (x_1 \sin \alpha - x_2 \cos \alpha)^2 &= x_1^2 + x_2^2 - (x_1 \cos \alpha + x_2 \sin \alpha)^2 \\ &= u_1 + 1 - (u_2 + 1)^2 \end{aligned}$$

We cannot, however, conclude that the equation $u_2 = 0$ is a consequence of $u_1 = 0$.

§ 46. *Solution of the equation $J = 1$.* Gauss has shown that two functions u, v satisfying the equation

$$\frac{\partial(u, v)}{\partial(x, y)} = 1$$

may be obtained by writing

$$v = \frac{\partial}{\partial u} F(x, u) \quad y = - \frac{\partial}{\partial x} F(x, u)$$

where F is an arbitrary function, and solving for u and v .

These equations show, in fact, that $v du - y dx$ is an exact differential; hence

$$\left(v \frac{\partial u}{\partial x} - y \right) dx + v \frac{\partial u}{\partial y} dy$$

is also an exact differential, and the condition

$$\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} - y \right) = 0$$

which must consequently be satisfied takes the required form

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 1$$

If (u, v) , (x, y) are regarded as the rectangular coordinates of two corresponding points P, Q in two planes W, Z respectively, and this correspondence can be defined in such a way that just one real point Q corresponds to a given real point P , then as P describes a closed curve C , Q will describe a closed curve Γ , provided F is a single-valued function of x and u .

The area of the curve Γ will, moreover, be equal to the area of the curve C if each of them is a simple oval.

To prove this we remark that the areas of C and Γ are represented by the integrals

$$\int v \, du \qquad \int y \, dx$$

taken round C and Γ respectively. Their difference is equal to

$$\int dF$$

and this vanishes, for when P is given Q is determined uniquely; consequently x and u are determined uniquely, and so F returns to its initial value after P and Q have made complete circuits round the curves C, Γ respectively.

It should be remarked that when $J = 1$ it follows that $v \, du - y \, dx$ is an exact differential dF , and so the solution of the equation $J = 1$ necessarily has the form given above.

The more general equation

$$\frac{\partial(u, v)}{\partial(x, y)} = f(x, y)$$

may be satisfied in a similar way by writing

$$v = \frac{\partial}{\partial u} F(x, u) \qquad \int f(x, y) dy = - \frac{\partial}{\partial x} F(x, u)$$

and solving for u and v .

EXAMPLES.

1. A correspondence is established between two points P and Q by making a tangent from P to an oval curve C equal and parallel to a tangent from Q to an oval curve Γ . Prove that in this correspondence areas are unaltered. Show also that the correspondence can be made one to one by supposing that each oval curve is described by a point in a definite direction and considering only one-half of the tangent at each point, namely, that part which would be described by the moving point if it left the curve.

2. Solve the equation $\frac{\partial(u, v, w)}{\partial(x, y, z)} = f(x, y, z)$

3. A point which moves in such a way that the quantities $f(x, y, z, t)$, $g(x, y, z, t)$ and $h(x, y, z, t)$ remain constant will always lie on the same moving surface of the family $k(x, y, z, t) = \text{const.}$ if the Jacobian

$$\frac{\partial(f, g, h, k)}{\partial(x, y, z, t)}$$

vanishes for each position of the moving point. In this example x, y, z denote the co-ordinates of a point and t the time.

4. A family of moving curves given by the equations $f(x, y, z, t) = a$, $g(x, y, z, t) = b$, where a and b are constants which vary from curve to curve, is such that each member continually intersects each one of a singly infinite set of moving curves belonging to a second family

$$\varphi(x, y, z, t) = \alpha \quad \chi(x, y, z, t) = \beta$$

Prove that the Jacobian $\frac{\partial(f, g, \varphi, \psi)}{\partial(x, y, z, t)}$

vanishes at all points (x, y, z, t) attained by the moving curves.

§ 47. *The derivation of a partial differential equation from a given primitive.* Let us suppose that we are given a relation

$$f(x, y, z, a, b) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

between x, y, z and two constants a, b . If x and y are regarded as independent variables, we obtain on differentiation

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \cdot \quad \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0 \quad . \quad . \quad . \quad (2)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. We now have three equations connecting x, y, z, p, q, a, b , and from these it is generally possible to eliminate the quantities a and b . The resulting equation

$$\varphi(x, y, z, p, q) = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

may be regarded as a *partial differential equation* for the determination of z . It is, moreover, an equation of the *first order*, because the first derivatives of z with regard to x and y are the highest derivatives that occur. We shall now show that the same partial differential equation can be derived from a different type of primitive.

If a and b are regarded as functions of x and y instead of constants, the equations obtained from (1) by differentiating with regard to x and y are

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial y} = 0$$

respectively. Now if the equations

$$\left. \begin{aligned} \frac{\partial f}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial x} &= 0 \\ \frac{\partial f}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial y} &= 0 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

are satisfied either identically or in consequence of (1), equations (2) will still be satisfied, and consequently z will again satisfy the partial differential equation (3).

Now equations (4) give either

$$\frac{\partial f}{\partial a} = 0 \quad \frac{\partial f}{\partial b} = 0 \quad \dots \dots \dots (5)$$

or
$$\frac{\partial(a, b)}{\partial(x, y)} = 0 \quad \dots \dots \dots (6)$$

In the first case the elimination of a and b from (5) (1) gives a relation

$$g(x, y, z) = 0 \quad \dots \dots \dots (7)$$

which can be regarded as a possible primitive of the partial differential equation.

In the second case it should be noticed that the relation (6) does not involve z ; it must, therefore, be satisfied identically and not in virtue of (1).

The relation implies that a and b are connected by a relation which does not involve x and y . If this relation is not simply $a = \text{const.}$, we may write

$$b = \psi(a) \quad \frac{\partial b}{\partial x} = \psi'(a) \frac{\partial a}{\partial x} \quad \frac{\partial b}{\partial y} = \psi'(a) \frac{\partial a}{\partial y}$$

and equations (4) can be replaced by the single relation

$$\frac{\partial f}{\partial a} + \psi'(a) \frac{\partial f}{\partial b} = 0 \quad \dots \dots \dots (8)$$

which must be satisfied either identically or in virtue of (1). If a is constant, equations (4) can be replaced by the single relation

$$\frac{\partial f}{\partial b} = 0 \quad \dots \dots \dots (8')$$

Eliminating a and b from equations (1) and (8) with the aid of the relation $b = \psi(a)$, we obtain a relation

$$G(x, y, z) = 0 \quad \dots \dots \dots (9)$$

which can be regarded as a possible primitive of the partial differential equation (3). If we write the relation between a and b in the more general form $F(a, b) = 0 \quad \dots \dots \dots (10)$

we find from (4) that
$$\frac{\partial(f, F)}{\partial(a, b)} = 0 \quad \dots \dots \dots (11)$$

and the equation $G(x, y, z) = 0$ may be regarded as the result of eliminating a and b from equations (1), (10), (11).

To distinguish between the different types of primitives we shall call (1) a *complete integral*, (7) a *singular integral* and (9) a *general integral*.

When a rule for finding all possible primitives of a partial differential equation has been found, we shall say that the *complete solution* has been obtained. The solution given by equation (7) is called a *singular integral* for a reason which will appear later; it should be noticed that it does not contain any arbitrary constants or arbitrary functions.

In this book we shall discuss only those methods of solution which are applicable to partial differential equations of the first order or to linear partial differential equations of the second order. It will be instructive, however, to give some examples on the formation of partial differential equations in which the equations are not necessarily of the types just mentioned. Some examples illustrating the derivation of systems of partial differential equations are also included. Sometimes the partial differential equation of a family of surfaces defined by some geometrical property is required. To obtain the required equation a certain amount of geometrical knowledge is frequently necessary; in this book only a few simple cases will be considered.

As an illustration let us find the partial differential equation of the family of all developable surfaces. We shall define a developable surface as the envelope of a singly infinite system of planes, that is, a system in which the coordinates of each member are functions of a single parameter.

Let the equation of one of the planes be

$$\xi x + \eta y + \zeta z + \tau = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where ξ, η, ζ, τ are functions of the parameter α . The envelope of the system is obtained by eliminating α between this equation and its derivative with regard to α , viz.

$$\frac{\partial \xi}{\partial \alpha} x + \frac{\partial \eta}{\partial \alpha} y + \frac{\partial \zeta}{\partial \alpha} z + \frac{\partial \tau}{\partial \alpha} = 0 \quad . \quad . \quad . \quad . \quad (13)$$

Let us write

$$\frac{\partial z}{\partial x} = p \quad \frac{\partial z}{\partial y} = q \quad \frac{\partial^2 z}{\partial x^2} = r \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad \frac{\partial^2 z}{\partial y^2} = t$$

then differentiating (12) with regard to x and y , we get

$$\xi + \zeta p = 0 \quad \eta + \zeta q = 0$$

Differentiating again, we find that

$$\left(\frac{\partial \xi}{\partial \alpha} + p \frac{\partial \zeta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial x} + \zeta r = 0$$

$$\left(\frac{\partial \xi}{\partial \alpha} + p \frac{\partial \zeta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial y} + \zeta s = 0$$

$$\left(\frac{\partial \eta}{\partial \alpha} + q \frac{\partial \zeta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial x} + \zeta s = 0$$

$$\left(\frac{\partial \eta}{\partial \alpha} + q \frac{\partial \zeta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial y} + \zeta t = 0$$

Hence

$$\zeta^2 (rt - s^2) = 0$$

The required partial differential equation is thus

$$rt - s^2 = 0$$

EXAMPLES.

1. Obtain a partial differential equation from the primitive

$$(x - a)^2 + (y - b)^2 + z^2 = 1$$

and show that $z^2 = 1$ is a singular solution.

2. Prove that the relation

$$f(x, y, z) + a g(x, y, z) + b h(x, y, z) = 0$$

gives rise to the partial differential equation

$$\begin{aligned} p \left[f \frac{\partial(g, h)}{\partial(y, z)} + g \frac{\partial(h, f)}{\partial(y, z)} + h \frac{\partial(f, g)}{\partial(y, z)} \right] \\ + q \left[f \frac{\partial(g, h)}{\partial(z, x)} + g \frac{\partial(h, f)}{\partial(z, x)} + h \frac{\partial(f, g)}{\partial(z, x)} \right] \\ = f \frac{\partial(g, h)}{\partial(x, y)} + g \frac{\partial(h, f)}{\partial(x, y)} + h \frac{\partial(f, g)}{\partial(x, y)} \end{aligned}$$

and show that the general integral can be written in the form

$$\frac{f(x, y, z)}{g(x, y, z)} = F \left[\frac{h(x, y, z)}{g(x, y, z)} \right]$$

where F is an arbitrary function.

3. Find the partial differential equations of the following families of surfaces :

1°. Surfaces of revolution.

2°. Cones.

3°. Surfaces generated by lines which meet the axis of z and are perpendicular to it.

4. Prove that if
- z
- is defined in terms of
- x
- and
- y
- by the equations

$$z = e^{-\gamma}(x \sin \alpha - y \cos \alpha) + u$$

$$0 = e^{-\gamma}(x \cos \alpha + y \sin \alpha) + \frac{\partial u}{\partial \alpha}$$

$$0 = e^{-\gamma}(y \cos \alpha - x \sin \alpha) + \frac{\partial u}{\partial \gamma}$$

where u is a solution of $\frac{\partial^2 u}{\partial \alpha^2} = \frac{\partial u}{\partial \gamma}$, then z satisfies the partial differential equation

$$rp^2 + 2spq + tq^2 = 0$$

5. Prove that if the elimination of an arbitrary function in a relation between
- x, y, z
- leads to an equation of type

$$u = f(v)$$

where u and v are functions of x, y, z, p, q , the elimination of the second arbitrary function f leads to a partial differential equation of type

$$rR + sS + tT + (rt - s^2)U = V$$

where R, S, T, U, V are certain functions of x, y, z . (Monge.)

6. Prove that if the relation $z = F(x, y)$ can be expressed in the form

$$f(x) + g(y) + h(z) = 0$$

$z = F(x, y)$ is a solution of the partial differential equation

$$\frac{\partial^2}{\partial x \partial y} \left(\log \frac{p}{q} \right) = 0 \quad (\text{P. de Saint Robert.})$$

7. Prove that if the relation $z = F(x, y)$ can be expressed in the form

$$\alpha(x)f(z) + \beta(y)g(z) + 1 = 0$$

$z = F(x, y)$ is a solution of the partial differential equations

$$\frac{\partial S}{\partial y} = 0 \quad \frac{\partial T}{\partial x} = 0$$

where S and T are defined by the equations

$$pqR = rq^2 - 2spq + tp^2$$

$$pV + qU = R$$

$$rV + sU + qS = \frac{\partial R}{\partial x}$$

$$sV + tU + pT = \frac{\partial R}{\partial y}$$

$$\frac{\partial s}{\partial x} V + \frac{\partial s}{\partial y} U + tS + rT = \frac{\partial^2 R}{\partial x \partial y} \quad (\text{Massau.})$$

8. Prove that the partial differential equations of the last example are equivalent to the two equations

$$\frac{1}{p} \frac{\partial w}{\partial x} = \frac{1}{q} \frac{\partial w}{\partial y} = v - uw$$

where

$$w = \frac{1}{v} \left[\frac{1}{q} \frac{\partial v}{\partial y} - \frac{1}{p} \frac{\partial v}{\partial x} \right]$$

$$v = \frac{1}{q} \frac{\partial u}{\partial y} - \frac{1}{p} \frac{\partial u}{\partial x}$$

$$u = \frac{s}{pq} \quad (\text{Lecornu.})$$

9. Prove that the function

$$V = F(x + iy, x_0 + iy_0) + G(x - iy, x_0 - iy_0) + C(xy_0 - x_0y)$$

where F and G are arbitrary functions and C an arbitrary constant, satisfies the partial differential equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \frac{\partial^2 V}{\partial x \partial x_0} + \frac{\partial^2 V}{\partial y \partial y_0} = 0 \quad \frac{\partial^2 V}{\partial x_0^2} + \frac{\partial^2 V}{\partial y_0^2} = 0$$

10. Prove that if $z = F[x + f(y)]$, the result of eliminating the arbitrary functions F and f is $qr = ps$

§ 48. *Geometrical representation of the different types of primitives.* Let (x, y, z) be regarded as the rectangular coordinates of a point in space, then the equation $f(x, y, z, a, b) = 0$ represents a family of surfaces, and $\phi(x, y, z, p, q) = 0$ may be regarded as a partial

differential equation expressing some property which is common to all the surfaces of the family. A surface which possesses this property will be called an *integral surface*.

When the quantities p, q are given by equations (2) and X, Y, Z are the coordinates of an arbitrary point, the equation

$$p(X - x) + q(Y - y) = Z - z$$

is the equation of the tangent plane at the point (x, y, z) of the surface $f = 0$.

If we regard p and q as quantities specifying the orientation of the tangent plane, it is clear from (3) that the tangent planes to the different integral surfaces through P have orientations which are restricted by a certain condition. This condition may be interpreted geometrically by saying that the planes are tangent planes of a certain cone C whose vertex is at the point (x, y, z) . It may happen, of course, that the condition implies that the planes all pass through a line, but in this case we regard the line as a degenerate cone. It may also happen that for some or even all values of x, y, z equation (3) breaks up into factors, each of which contains at least one of the quantities p, q . In this case the planes are tangent planes to more than one cone, but we can regard the aggregate of all these cones as the cone C .

Let us now suppose that each tangent plane to the cone C associated with the point (x, y, z) is a tangent plane at this point to at least one surface of the family (1). We shall call this *supposition A*.

In this case if a relation

$$F(x, y, z) = 0 \quad (12)$$

gives a solution of the partial differential equation $\varphi(x, y, z, p, q) = 0$, the tangent plane at a point (x, y, z) of the integral surface $F = 0$ must be a tangent plane of the cone C , and consequently a tangent plane at (x, y, z) to at least one surface of the family $f(x, y, z, a, b) = 0$. This means either that the surface $F = 0$ belongs to the family or that it is an envelope of surfaces belonging to the family.

In the latter case it may be the envelope of a singly infinite set of surfaces belonging to the family or of a doubly infinite set. A singly infinite set is obtained by making b a function of a in the equation $f(x, y, z, a, b) = 0$. This equation may then be regarded as an equation determining the possible values of a when the point (x, y, z) is given, and when this point lies on the surface $F = 0$ an equation defining a as a function of x and y may be obtained by eliminating z from the equations

$$f(x, y, z, a, b) = 0 \quad F(x, y, z) = 0$$

If $b = \psi(a)$ is the relation between a and b , the condition that a surface of the associated family should have the same tangent plane at (x, y, z) as the surface $F = 0$ is simply expressed by equation (8),

from which equations (4) may be derived. To see this we remark that when a is expressed in terms of x and y by the process described above and then substituted in the equation $f(x, y, z, a, b) = 0$, the resulting equation is satisfied in virtue of $F = 0$, and must consequently be equivalent to the equation $F = 0$.

Now, equations (4) imply that the values of p and q obtained from the modified equation (1) are the same as those given by (2); in other words, the tangent plane at the point (x, y, z) to the surface $F = 0$ is the same as the tangent plane at this point to a surface of the singly infinite system. Equation (8) is satisfied in virtue of the equations $F = 0$, $b = \psi(a)$ and $f = 0$; hence we may regard the equation $F = 0$ as a consequence of equations (1), (8) and $b = \psi(a)$. This means that $F = 0$ is a factor of the equation obtained by eliminating a and b from the equations just mentioned or is equivalent to an equation derived from one of the factors.

It should be remarked that the eliminant may contain factors representing surfaces which are not envelopes of integral surfaces. The locus of double curves on integral surfaces belonging to the singly infinite family may appear as a factor of the eliminant, likewise the locus of cuspidal edges. The surfaces representing these loci are not generally integral surfaces; in other words, certain factors of the eliminant may not provide solutions of the partial differential equation. With special forms of the function $\psi(a)$ it may happen indeed that the equations

$$f = 0 \quad b = \psi(a) \quad \frac{\partial f}{\partial a} + \psi'(a) \frac{\partial f}{\partial b} = 0$$

do not lead to a solution of the partial differential equation at all.

To illustrate this let us consider the equation

$$f \equiv (y + bz)^2 - (x + a)^3 = 0$$

which can be regarded as a complete integral of the partial differential equation

$$27q^2(qy - z) + 8p^3 = 0$$

If we put $b = 0$ and form the a -discriminant, we get simply

$$y^2 = 0$$

hence in this case the method does not provide us with a solution of the differential equation. The surface $y = 0$ is a locus of cuspidal curves.

If an integral surface $F = 0$ is an envelope of a doubly infinite system of integral surfaces belonging to the family $f(x, y, z, a, b) = 0$, there will generally be an infinite number of these surfaces through a point (x, y, z) of the envelope which have the same tangent plane at this point.

Let us select a limited number of these by making a an arbitrarily chosen function of x and y , then if $F = 0$ is not a particular surface of the family (1), we may eliminate z from the equations $f = 0$,

$F = 0$ and express b as a function of x and y . Moreover, b will not be simply a function of a , consequently equations (4) which state that the tangent planes at the point (x, y, z) to the surfaces $f = 0$, $F = 0$ are the same can only be satisfied if

$$\frac{\partial f}{\partial a} = 0 \quad \frac{\partial f}{\partial b} = 0$$

Consequently an envelope of a doubly infinite system of surfaces belonging to the family (1) is obtained when it exists by eliminating a and b from the equations

$$f = 0 \quad \frac{\partial f}{\partial a} = 0 \quad \frac{\partial f}{\partial b} = 0$$

The eliminant from these equations gives the *general envelope* of the family (1) when such an envelope exists, and this envelope may consist of either one or a number of surfaces, each of which provides us with a *singular solution* of the partial differential equation, assuming as we have done that an enveloping surface does not belong to the family (1).

The eliminant may also contain other factors which do not generally provide us with solutions of the partial differential equation. The loci of conical points, double curves, cuspidal curves and more complicated singularities may appear as factors. For a full discussion of the different possibilities the reader is referred to a memoir by Prof. M. J. M. Hill.* The isoclinal surfaces which correspond to the isoclinal curves of § 39 are considered by H. Levy.†

Some interesting theorems are given by Darboux.‡

It should be remarked that the preceding method may fail to give a singular solution when the function f is not a single-valued function of its arguments. Let us consider, for instance, the case when

$$f(x, y, z, a, b) \equiv y + 2(z - x - y)^{\frac{1}{2}} + a(2y - z) + b = 0$$

$$\text{The equations} \quad \frac{\partial f}{\partial a} = 0 \quad \frac{\partial f}{\partial b} = 0 \quad f = 0$$

clearly cannot be satisfied simultaneously. The associated partial differential equation

$$\{1 + (z - x - y)^{\frac{1}{2}}\}p + q = 2$$

however, possesses the solution $z = x + y$, which is not included in the above complete integral and is not given by the general integral

$$y + 2(z - x - y)^{\frac{1}{2}} = F(2y - z)$$

It should be noticed, however, that if we rationalise the complete integral and write it in the form

$$[y + a(2y - z) + b]^2 = 4(z - x - y)$$

* *Phil. Trans. A.* (1892), pp. 141-278.

† *Proc. Roy. Soc. Edinburgh*, Vol. 32 (1911-12), p. 150.

‡ *Mémoires présentés à l'Académie des Sciences*, t. 27.

the method indicates that $z = x + y$ is a *singular integral*, for it is clearly an envelope of integral surfaces.

Summing up our results we can say that when supposition A is valid, a solution of the partial differential equation $\varphi = 0$ generally belongs to one of three classes of integrals which comprise integrals of the following types :

1°. Integrals obtained by giving particular constant values to a and b in the complete integral $f(x, y, z, a, b) = 0$.

2°. Integrals obtained by choosing particular forms of the arbitrary function $F(a, b)$ in the general integral defined by the equations

$$f = 0 \quad F = 0 \quad \frac{\partial(f, F)}{\partial(a, b)} = 0$$

3°. Singular integrals obtained by eliminating a and b from the equations

$$f = 0 \quad \frac{\partial f}{\partial a} = 0 \quad \frac{\partial f}{\partial b} = 0$$

An integral which does not belong to one of these classes is called *special*. A special integral may, however, be a limiting case of a complete integral. For instance, in the case of the equation *

$$px - qy = (z - xy)^2$$

a complete integral is $\frac{1}{xe^{z-xy}} = \frac{xy + a}{xy + b}$

and the integral $z - xy$ is regarded as special because it is not possible to assign *finite* values to a and b in the equations

$$p = y + (z - xy)^2 \left[\frac{1}{x} - \frac{y}{xy + a} + \frac{y}{xy + b} \right]$$

$$q = x + (z - xy)^2 \left[-\frac{x}{xy + a} + \frac{x}{xy + b} \right]$$

so that we may have $p = y$, $q = x$. The equation $z - xy = 0$ may, however, be derived from the complete integral by making either a or b infinite or equal to $-xy$.

The case in which supposition A is not valid is rather difficult to deal with. If the equation $\varphi = 0$ breaks up into factors, each of which contains derivatives of z , there will generally be more than one type of general integral. It may, however, be argued that the *composite* partial differential equation is not the true partial differential equation corresponding to a single-valued complete integral $f = 0$ which gives rise to only one type of general integral. This argument is valid if we can justify the rejection of certain factors in the process of elimination by which the partial differential equation $\varphi = 0$ is derived.

* This example is given by A. R. Forsyth, *Differential Equations*, 4th edition (1914), p. 383.

To illustrate this point let us consider the primitive

$$f(x, y, z, a, b) \equiv (z + by)^3 - (x + a)^2 = 0$$

To form the corresponding partial differential equation $\varphi = 0$ we must eliminate a and b from the preceding equation and the two equations

$$3p(z + by)^2 = 2(x + a) \quad 3(q + b)(z + by)^2 = 0$$

The result may be written in the form

$$[9p^2(z - qy) - 4](z - qy)^3 = 0$$

The equation $9p^2(z - qy) - 4 = 0$ must be regarded as the *true* partial differential equation corresponding to the primitive $f = 0$. The equation $z - qy = 0$ can be integrated like an ordinary differential equation, and is seen to possess the general integral $z = y\psi(x)$. A complete integral $z = y(a + bx)$ may be derived from this by choosing a particular form of ψ . This complete integral is essentially different from $f = 0$, because the two complete integrals lead to general integrals which are not equivalent.

It is natural to ask if a partial differential equation can possess more than one type of complete integral leading to the same type of general integral. The answer is that this is quite possible; for instance, in the case of the equation $z - qy = 0$, the two complete integrals

$$z = y(a + bx) \quad z = y(a + bx^2)$$

lead to the same type of general integral, viz. $z = y\psi(x)$.

A similar remark holds in the case of the general equation, for if in the process of finding the general integral we write $b = \psi(a, \alpha, \beta)$, where α and β are arbitrary constants, the integral so obtained can be regarded as a complete integral in which α and β are the arbitrary constants.

To see that this new complete integral leads to the same general integral as before, we write $\beta = \chi(\alpha)$ and differentiate the equation $f = 0$ with regard to α . This gives

$$\frac{\partial f}{\partial b} \left[\frac{\partial b}{\partial \alpha} + \frac{\partial b}{\partial \beta} \chi'(\alpha) \right] = 0$$

The equations

$$\beta = \chi(\alpha) \quad \frac{\partial b}{\partial \alpha} + \frac{\partial b}{\partial \beta} \chi'(\alpha) = 0$$

may now be used to express α and β in terms of a , and then the equation $b = \psi(a, \alpha, \beta)$ takes the form $b = \omega(a)$. Also since

$$\frac{\partial b}{\partial \alpha} + \frac{\partial b}{\partial \beta} \chi'(\alpha) = 0$$

we have

$$\frac{d\omega}{da} = \frac{\partial \psi}{\partial a}$$

hence the equation

$$\frac{\partial f}{\partial a} + \frac{\partial \psi}{\partial a} \frac{\partial f}{\partial b} = 0$$

may be written in the form

$$\frac{\partial f}{\partial a} + \frac{d\omega}{da} \frac{\partial f}{\partial b} = 0$$

and so it follows that the general integral derived from the new complete integral is the same as that derived from the old.

EXAMPLES.

1. If $\varphi(x, y, z, p, q)$ becomes $\chi(x, y, z)$ when p and q are expressed in terms of x, y and z with the aid of the equations $\frac{\partial \varphi}{\partial p} = 0, \frac{\partial \varphi}{\partial q} = 0$, the partial differential equation

$$\varphi(x, y, z, p, q) - \chi(x, y, z) = 0$$

has the peculiarity that the equations to determine a singular solution can be satisfied in an infinite number of ways. (G. Chrystal.)

2. Discuss the equation

$$(px + qy - z)^2 - p^2 - q^2 = \frac{x^2}{1 - x^2 - y^2}$$

the general solution of which is $\varphi(a, b) = 0$, where $z = a\sqrt{x^2 + y^2 - 1}$ and $x \cos b + y \sin b = 1$. (G. Chrystal.)

§ 49. *The characteristics of a partial differential equation of the first order.* The envelope of a singly infinite system of integral surfaces can be regarded as the locus of the ultimate curves of intersection of consecutive surfaces of the system.

Let $(a, b), (a + \delta a, b + \delta b)$ be the parameters of two consecutive surfaces, then $\delta a, \delta b$ are small quantities which will be made to approach zero. If the derivatives of f with respect to a and b are finite or determinate, the equations

$$f(x, y, z, a, b) = 0 \quad f(x, y, z, a + \delta a, b + \delta b) = 0$$

give $f_a(x, y, z, a + \theta \delta a, b + \delta b) \delta a + f_b(x, y, z, a, b + \omega \delta b) \delta b = 0$ when we apply the mean value theorem, θ and ω being quantities lying between 0 and 1, and f_a, f_b denoting the functions derived from f by differentiating with regard to a and b respectively. Also, if $b = \psi(a)$, where $\psi(a)$ is a continuous function with a determinate derivative, we have $\delta b = \delta a \psi'(a + \eta \delta a)$, where $0 < \eta < 1$; hence $f_a(x, y, z, a + \theta \delta a, b + \delta b) + \psi'(a + \eta \delta a) f_b(x, y, z, a, b + \omega \delta b) = 0$

Now make δa and δb tend to zero, then we find that the ultimate curve of intersection of the surfaces

$$f(x, y, z, a, b) = 0 \quad \text{and} \quad f(x, y, z, a + \delta a, b + \delta b) = 0$$

is given by the equations $f(x, y, z, a, b) = 0$

$$b = \psi(a)$$

$$\frac{\partial f}{\partial a} + \psi'(a) \frac{\partial f}{\partial b} = 0$$

This curve is called a *characteristic* of the envelope.

As a point moves along a characteristic a and b remain constant, but the quantities x, y, z, p, q vary, their increments being connected by the equations $dz = p dx + q dy$

$$\begin{aligned} & \left[\frac{\partial^2 f}{\partial a \partial x} + \psi'(a) \frac{\partial^2 f}{\partial b \partial x} \right] dx + \left[\frac{\partial^2 f}{\partial a \partial y} + \psi'(a) \frac{\partial^2 f}{\partial b \partial y} \right] dy \\ & \quad + \left[\frac{\partial^2 f}{\partial a \partial z} + \psi'(a) \frac{\partial^2 f}{\partial b \partial z} \right] dz = 0 \\ & \left[\frac{\partial^2 f}{\partial x^2} + p \frac{\partial^2 f}{\partial x \partial z} \right] dx + \left[\frac{\partial^2 f}{\partial x \partial y} + p \frac{\partial^2 f}{\partial y \partial z} \right] dy \\ & \quad + \left[\frac{\partial^2 f}{\partial x \partial z} + p \frac{\partial^2 f}{\partial z^2} \right] dz + \frac{\partial f}{\partial z} dp = 0 \\ & \left[\frac{\partial^2 f}{\partial x \partial y} + q \frac{\partial^2 f}{\partial x \partial z} \right] dx + \left[\frac{\partial^2 f}{\partial y^2} + q \frac{\partial^2 f}{\partial y \partial z} \right] dy \\ & \quad + \left[\frac{\partial^2 f}{\partial y \partial z} + q \frac{\partial^2 f}{\partial z^2} \right] dz + \frac{\partial f}{\partial z} dq = 0 \end{aligned}$$

the last three of which are derived from the equations

$$\frac{\partial f}{\partial a} + \psi'(a) \frac{\partial f}{\partial b} = 0 \quad \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0$$

respectively. We shall now show that it is possible to express the ratios of dx, dy, dz, dp, dq in terms of the derivatives of the function $\varphi(x, y, z, p, q)$ occurring in the partial differential equation satisfied by $f(x, y, z, a, b)$.

Since $\varphi = 0$ is a consequence of

$$f = 0 \quad \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0$$

the equation

$$\frac{\partial \varphi}{\partial x} \delta x + \frac{\partial \varphi}{\partial y} \delta y + \frac{\partial \varphi}{\partial z} \delta z + \frac{\partial \varphi}{\partial p} \delta p + \frac{\partial \varphi}{\partial q} \delta q = 0 \quad (1)$$

must be a consequence of the equations

$$\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial a} \delta a + \frac{\partial f}{\partial b} \delta b = 0 \quad (2)$$

$$\begin{aligned} & \left(\frac{\partial^2 f}{\partial x^2} + p \frac{\partial^2 f}{\partial x \partial z} \right) \delta x + \left(\frac{\partial^2 f}{\partial x \partial y} + p \frac{\partial^2 f}{\partial y \partial z} \right) \delta y + \left(\frac{\partial^2 f}{\partial x \partial z} + p \frac{\partial^2 f}{\partial z^2} \right) \delta z \\ & \quad + \frac{\partial f}{\partial z} \delta p + \left(\frac{\partial^2 f}{\partial a \partial x} + p \frac{\partial^2 f}{\partial a \partial z} \right) \delta a + \left(\frac{\partial^2 f}{\partial b \partial x} + p \frac{\partial^2 f}{\partial b \partial z} \right) \delta b = 0 \quad (3) \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial^2 f}{\partial x \partial y} + q \frac{\partial^2 f}{\partial x \partial z} \right) \delta x + \left(\frac{\partial^2 f}{\partial y^2} + q \frac{\partial^2 f}{\partial y \partial z} \right) \delta y + \left(\frac{\partial^2 f}{\partial y \partial z} + q \frac{\partial^2 f}{\partial z^2} \right) \delta z \\ & \quad + \frac{\partial f}{\partial z} \delta q + \left(\frac{\partial^2 f}{\partial a \partial y} + q \frac{\partial^2 f}{\partial a \partial z} \right) \delta a + \left(\frac{\partial^2 f}{\partial b \partial y} + q \frac{\partial^2 f}{\partial b \partial z} \right) \delta b = 0 \quad (4) \end{aligned}$$

Eliminating δp and δq from (1), (3) and (4), we obtain an equation which must be a consequence of (2); consequently

$$\begin{aligned} \frac{\partial \varphi}{\partial p} \left(\frac{\partial^2 f}{\partial x^2} + p \frac{\partial^2 f}{\partial x \partial z} \right) + \frac{\partial \varphi}{\partial q} \left(\frac{\partial^2 f}{\partial x \partial y} + q \frac{\partial^2 f}{\partial x \partial z} \right) - \frac{\partial \varphi}{\partial x} \frac{\partial f}{\partial z} &= \lambda \frac{\partial f}{\partial x} \\ \frac{\partial \varphi}{\partial p} \left(\frac{\partial^2 f}{\partial x \partial y} + p \frac{\partial^2 f}{\partial y \partial z} \right) + \frac{\partial \varphi}{\partial q} \left(\frac{\partial^2 f}{\partial y^2} + q \frac{\partial^2 f}{\partial y \partial z} \right) - \frac{\partial \varphi}{\partial y} \frac{\partial f}{\partial z} &= \lambda \frac{\partial f}{\partial y} \\ \frac{\partial \varphi}{\partial p} \left(\frac{\partial^2 f}{\partial x \partial z} + p \frac{\partial^2 f}{\partial z^2} \right) + \frac{\partial \varphi}{\partial q} \left(\frac{\partial^2 f}{\partial y \partial z} + q \frac{\partial^2 f}{\partial z^2} \right) - \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial z} &= \lambda \frac{\partial f}{\partial z} \\ \frac{\partial \varphi}{\partial p} \left(\frac{\partial^2 f}{\partial a \partial x} + p \frac{\partial^2 f}{\partial a \partial z} \right) + \frac{\partial \varphi}{\partial q} \left(\frac{\partial^2 f}{\partial a \partial y} + q \frac{\partial^2 f}{\partial a \partial z} \right) &= \lambda \frac{\partial f}{\partial a} \\ \frac{\partial \varphi}{\partial p} \left(\frac{\partial^2 f}{\partial b \partial x} + p \frac{\partial^2 f}{\partial b \partial z} \right) + \frac{\partial \varphi}{\partial q} \left(\frac{\partial^2 f}{\partial b \partial y} + q \frac{\partial^2 f}{\partial b \partial z} \right) &= \lambda \frac{\partial f}{\partial b} \end{aligned}$$

Eliminating the unknown quantity λ from these equations with the aid of the relations

$$\frac{\partial f}{\partial a} + \psi'(a) \frac{\partial f}{\partial b} = 0 \quad \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0$$

we find that

$$\begin{aligned} &\left[\frac{\partial^2 f}{\partial a \partial x} + p \frac{\partial^2 f}{\partial a \partial z} + \psi'(a) \frac{\partial^2 f}{\partial b \partial x} + p \psi'(a) \frac{\partial^2 f}{\partial b \partial z} \right] \frac{\partial \varphi}{\partial p} \\ &+ \left[\frac{\partial^2 f}{\partial a \partial y} + q \frac{\partial^2 f}{\partial a \partial z} + \psi'(a) \frac{\partial^2 f}{\partial b \partial y} + q \psi'(a) \frac{\partial^2 f}{\partial b \partial z} \right] \frac{\partial \varphi}{\partial q} = 0 \\ &\left[\frac{\partial^2 f}{\partial x^2} + 2p \frac{\partial^2 f}{\partial x \partial z} + p^2 \frac{\partial^2 f}{\partial z^2} \right] \frac{\partial \varphi}{\partial p} \\ &+ \left[\frac{\partial^2 f}{\partial x \partial y} + p \frac{\partial^2 f}{\partial y \partial z} + q \frac{\partial^2 f}{\partial x \partial z} + pq \frac{\partial^2 f}{\partial z^2} \right] \frac{\partial \varphi}{\partial q} \\ &= \frac{\partial f}{\partial z} \left[\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z} \right] \\ &\left[\frac{\partial^2 f}{\partial x \partial y} + p \frac{\partial^2 f}{\partial y \partial z} + q \frac{\partial^2 f}{\partial x \partial z} + pq \frac{\partial^2 f}{\partial z^2} \right] \frac{\partial \varphi}{\partial p} \\ &+ \left[\frac{\partial^2 f}{\partial y^2} + 2q \frac{\partial^2 f}{\partial y \partial z} + q^2 \frac{\partial^2 f}{\partial z^2} \right] \frac{\partial \varphi}{\partial q} = \frac{\partial f}{\partial z} \left[\frac{\partial \varphi}{\partial y} + q \frac{\partial \varphi}{\partial z} \right] \end{aligned}$$

Comparing these equations with those which give the ratios of dx , dy , dz , dp , dq , we find that

$$\frac{dx}{\frac{\partial \varphi}{\partial p}} = \frac{dy}{\frac{\partial \varphi}{\partial q}} = \frac{dz}{p \frac{\partial \varphi}{\partial p} + q \frac{\partial \varphi}{\partial q}} = - \frac{dp}{\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z}} = - \frac{dq}{\frac{\partial \varphi}{\partial y} + q \frac{\partial \varphi}{\partial z}}$$

These are called *the differential equations of the characteristics*.

A characteristic may be defined, in fact, as the aggregate of elements x, y, z, p, q common to two consecutive integral surfaces, an element being understood to mean a point (x, y, z) and an associated plane through the point, the orientation of the plane being specified by the quantities p, q .

If the point (x, y, z) lies on a general envelope of the family of integral surface $f(x, y, z, a, b) = 0$, there will be ∞' integral surfaces having the common element (x, y, z, p, q) , and consequently in general there will be ∞' characteristics having this common element. These characteristics will usually pass through the point (x, y, z) in different directions, and so a knowledge of x, y, z, p, q will not be sufficient to determine the ratios of dx, dy, dz, dp, dq for a characteristic with this element ; consequently we must have

$$\frac{\partial \varphi}{\partial p} = 0 \quad \frac{\partial \varphi}{\partial q} = 0 \quad \frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z} = 0 \quad \frac{\partial \varphi}{\partial y} + q \frac{\partial \varphi}{\partial z} = 0$$

It follows from this result that a singular solution of the partial differential equation $\varphi = 0$ may be derived directly from the differential equation by eliminating p and q from the equations

$$\varphi = 0 \quad \frac{\partial \varphi}{\partial p} = 0 \quad \frac{\partial \varphi}{\partial q} = 0$$

and retaining only factors of the eliminant which can also be derived from the equations

$$\varphi = 0 \quad \frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z} = 0 \quad \frac{\partial \varphi}{\partial y} + q \frac{\partial \varphi}{\partial z} = 0$$

The complete eliminant from the first set of three equations may include tac-loci, loci of pinch-points, cuspidal curves, etc., which do not provide us with solutions of the partial differential equation.

§ 50. *Lagrange's Linear Equation.* Let φ be an arbitrary function of two quantities u, v which are themselves functions of x, y, z ; then the elimination of the arbitrary function φ from the equation

$$\varphi(u, v) = 0 \quad \dots \dots \dots (1)$$

leads to a simple type of partial differential equation. To see this we differentiate with regard to x and y , and eliminate the derivatives of φ from the resulting equations

$$\begin{aligned} \frac{\partial \varphi}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial \varphi}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) &= 0 \\ \frac{\partial \varphi}{\partial u} \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) + \frac{\partial \varphi}{\partial v} \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) &= 0 \end{aligned}$$

The eliminant

$$\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) = \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right)$$

can be written in the form

$$\bullet \quad Pp + Qq = R \quad \dots \dots \dots (2)$$

where $P = \lambda \frac{\partial(u, v)}{\partial(y, z)} \quad Q = \lambda \frac{\partial(u, v)}{\partial(z, x)} \quad R = \lambda \frac{\partial(u, v)}{\partial(x, y)}$

and λ is an arbitrary multiplier. If λ is independent of p and q , the resulting partial differential equation is said to be *linear* because the left-hand side is a polynomial of the first degree in p and q .

Now suppose that a partial differential equation of this type is given. To solve it we must endeavour to find a suitable pair of functions u and v . Let a and b be arbitrary constants; then the equations $u = a$, $v = b$ should give particular solutions of the partial differential equation. We may regard them as the equations of two particular integral surfaces.

Now let a point (x, y, z) move along the curve of intersection of these two surfaces; then the increments dx , dy , dz are connected by the equations

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

consequently the curve is given by *Lagrange's auxiliary equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \dots \dots \dots (3)$$

The curve is, in fact, a characteristic of the partial differential equation. To determine a suitable pair of functions u and v we have simply to obtain two independent integrals of the equations (3). These may be regarded as two simultaneous equations for determining x and y as functions of z . In solving these equations it should be noted that when one solution $u = a$ has been obtained we may use this result to eliminate one of the variables and obtain an ordinary differential equation connecting the other two. If $v = b$ is a solution of this last equation, the general integral of the partial differential equation is given by (1). To verify that the partial differential equation is satisfied, we have simply to retrace the steps whereby (3) was obtained.

In connection with a system of differential equations of type (3) there is an important theorem due to Jacobi which is sometimes useful.

Let us write $\lambda = \frac{1}{M}$; then we have

$$MP = \frac{\partial(u, v)}{\partial(y, z)} \quad MQ = \frac{\partial(u, v)}{\partial(z, x)} \quad MR = \frac{\partial(u, v)}{\partial(x, y)}$$

consequently M satisfies the partial differential equation

$$\frac{\partial}{\partial x}(MP) + \frac{\partial}{\partial y}(MQ) + \frac{\partial}{\partial z}(MR) = 0$$

Let us suppose that M and N are two distinct solutions of this equation, and write $N = M\theta$; then

$$\frac{\partial}{\partial x}(M\theta P) + \frac{\partial}{\partial y}(M\theta Q) + \frac{\partial}{\partial z}(M\theta R) = 0$$

and consequently
$$MP \frac{\partial \theta}{\partial x} + MQ \frac{\partial \theta}{\partial y} + MR \frac{\partial \theta}{\partial z} = 0$$

On substituting for MP , MQ , MR , this equation takes the form

$$\frac{\partial(\theta, u, v)}{\partial(x, y, z)} = 0$$

This equation is satisfied identically and not in virtue of the equations $u = a$, $v = b$; hence θ must be a function of u and v .

Quantities such as M and N are called *multipliers*; the theorem which has just been proved indicates that the ratio of any two multipliers is a solution of the partial differential equation (1) and of the system of equations (3).

An important property of a multiplier is that when one integral of the equations (3) has been obtained and when any multiplier is known, another integral can be obtained by quadratures only.

Let the integral which has been obtained by

$$u(x, y, z) = a$$

then, pursuing the method which has already been sketched, let one of the variables (say z) be eliminated with the aid of this equation.

If $v(x, y, z) = b$ is a second solution of the equations (3), the modified form of the equation $Qdx = Pdy$ will be satisfied by the relation $G(x, y, a) = b$, which is obtained by eliminating z from the equations $u = a$, $v = b$.

Now
$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial G}{\partial x} + \frac{\partial G}{\partial a} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial G}{\partial y} + \frac{\partial G}{\partial a} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} &= \frac{\partial G}{\partial a} \frac{\partial u}{\partial z} \end{aligned}$$

consequently
$$MP = \frac{\partial(u, v)}{\partial(y, z)} = - \frac{\partial G}{\partial y} \frac{\partial u}{\partial z}$$

$$MQ = \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial G}{\partial x} \frac{\partial u}{\partial z}$$

Hence
$$\begin{aligned} dG &\equiv \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy \\ &= \frac{M}{\frac{\partial u}{\partial z}} (Q dx - P dy) \end{aligned}$$

A second integral of the equations (3) is thus given by the relation

$$\int \frac{M}{\partial u} (Q dx - P dy) = b$$

If
$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

we may evidently take $M = 1$.

All the preceding results may be generalised by considering the partial differential equation derived from a relation of type

$$\varphi(u_1, u_2, \dots u_n) = 0 \quad \dots \dots \dots (I)$$

where φ is an arbitrary function and $u_1, u_2, \dots u_n$ particular functions of n independent variables $x_1, x_2, \dots x_n$ and a dependent variable z . To form the partial differential equation we differentiate with respect to x_s . The resulting equation

$$\frac{\partial \varphi}{\partial u_1} \left(\frac{\partial u_1}{\partial x_s} + \frac{\partial u_1}{\partial z} \frac{\partial z}{\partial x_s} \right) + \dots \frac{\partial \varphi}{\partial u_n} \left(\frac{\partial u_n}{\partial x_s} + \frac{\partial u_n}{\partial z} \frac{\partial z}{\partial x_s} \right) = 0$$

may be written in the form

$$\frac{\partial \varphi}{\partial u_1} \frac{\partial u_1}{\partial x_s} + \frac{\partial \varphi}{\partial u_2} \frac{\partial u_2}{\partial x_s} + \dots \frac{\partial \varphi}{\partial u_n} \frac{\partial u_n}{\partial x_s} + K \frac{\partial z}{\partial x_s} = 0$$

where
$$\frac{\partial \varphi}{\partial u_1} \frac{\partial u_1}{\partial z} + \frac{\partial \varphi}{\partial u_2} \frac{\partial u_2}{\partial z} + \dots \frac{\partial \varphi}{\partial u_n} \frac{\partial u_n}{\partial z} - K = 0$$

Putting $s = 1, 2, \dots n$ in turn and eliminating $\frac{\partial \varphi}{\partial u_1}, \frac{\partial \varphi}{\partial u_2}, \dots K$, we obtain the partial differential equation

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_1} & \frac{\partial z}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_2} & \frac{\partial z}{\partial x_2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \dots & \frac{\partial u_n}{\partial z} & -1 \end{vmatrix} = 0$$

This may be written in the form

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \dots P_n \frac{\partial z}{\partial x_n} = R \quad \dots \dots \dots (II)$$

and is said to be *linear*. When the partial differential equation is given, we may obtain a suitable set of functions $u_1 \dots u_n$ by solving the system of simultaneous equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots \frac{dx_n}{P_n} = \frac{dz}{R} \quad \dots \dots \dots (III)$$

A quantity M , which is such that functions $u_1, u_2, \dots u_n$ exist for which

$$MP_1 = \frac{\partial(u_1, u_2, \dots u_n)}{\partial(x_2, x_3, \dots x_n, z)}$$

$$\dots \dots \dots$$

$$MR = \frac{\partial(u_1, u_2, \dots u_n)}{\partial(x_1, x_2, \dots x_n)}$$

is called a *multiplier*, it satisfies the partial differential equation

$$\frac{\partial}{\partial x_1}(MP_1) + \frac{\partial}{\partial x_2}(MP_2) + \dots \frac{\partial}{\partial x_n}(MP_n) + \frac{\partial}{\partial z}(MR) = 0$$

The ratio of two different multipliers is a solution of the partial differential equation (II) and of the system of differential equations (III).

If $u_1 = \alpha_1, u_2 = \alpha_2, \dots u_{n-1} = \alpha_{n-1}$ be $n - 1$ integrals of the equations just mentioned, and M is any multiplier, a last integral of the equations is given by

$$\int \frac{M}{\partial(u_1, u_2, \dots u_{n-1})} (P_1 dx_2 - P_2 dx_1) = \alpha_n$$

where α_n is an arbitrary constant, and all the quantities in the subject of integration are to be expressed in terms of $x_1, x_2, \alpha_1, \dots \alpha_{n-1}$ by means of the $n - 1$ known integrals of the system (III).

The quantity M was called by Jacobi a *last multiplier*.

Equations of types (II) and (III) occur very frequently in mechanical and hydrodynamical problems.

Let us suppose that a swarm of particles is in motion, and that the component velocities (u, v, w) of the particle which is at the point (x, y, z) at time t are known functions of x, y, z, t . A moving surface whose equation at time t is represented by

$$f(x, y, z) = t$$

will always contain the same system of particles if for an increment δt

$$f(x + u \delta t, y + v \delta t, z + w \delta t) = t + \delta t$$

that is, if f satisfies the partial differential equation

$$u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} = 1$$

The paths of the individual particles are determined by the system of auxiliary equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dt}{1}$$

If a function $\rho(x, y, z, t)$ representing the density of the swarm of

particles exists, it is a multiplier of the preceding equations if the *equation of continuity*

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0$$

is satisfied.

Example. Consider the equation

$$(y^2 + z^2 - x^2) \frac{\partial z}{\partial x} - 2xy \frac{\partial z}{\partial y} + 2xz = 0$$

Lagrange's auxiliary equations are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

A first solution is derived from the relation $\frac{dy}{y} = \frac{dz}{z}$, and may be written in the form $z = ay$. Modifying the differential equations with the aid of this relation, we find that

$$\frac{dx}{y^2(1 + a^2) - x^2} = \frac{dy}{-2xy}$$

This is a homogeneous equation; to solve it we write $x = ty$: the equation then becomes

$$\frac{2t dt}{1 + a^2 + t^2} + \frac{dy}{y} = 0$$

$$\therefore y(1 + a^2 + t^2) = b$$

or

$$x^2 + y^2(1 + a^2) = by$$

Thus

$$a = \frac{z}{y} \quad b = \frac{x^2 + y^2 + z^2}{y}$$

and so a general solution of the partial differential equation is given by

$$F\left(\frac{z}{y}, \frac{x^2 + y^2 + z^2}{y}\right) = 0$$

where F is an arbitrary function.

In the following examples p and q are written for $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ respectively, while $p_1, p_2, p_3, \dots$ denote $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}, \dots$.

EXAMPLES.

1. Solve the equations

$$ap + bq = 1 \quad p(x - a) + q(y - b) = z - c \quad py - qx = 0$$

where a, b, c are constants, and interpret the results geometrically.

2. Solve the equations

$$\begin{aligned} p + zq &= 0 & py - qx &= 1 \\ xzp + yzq &= xy & x^2p - xyq + y^2 &= 0 \end{aligned}$$

3. Solve the equation

$$(x_1 + x_2 + z)p_1 + (x_1 + x_1 + z)p_2 + (x_1 + x_2 + z)p_3 = x_1 + x_2 + x_3$$

4. Particles are projected from a given point with various velocities in various directions; the particles move under gravity without resistance from the air. Prove that if a moving surface $t = f(x, y, z)$ always contains the same particles, f must satisfy the partial differential equation

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + (z - \frac{1}{2}gt^2) \frac{\partial f}{\partial z} = f$$

where g is the acceleration due to gravity. Show also that $M = f^{-4}$ is a multiplier of Lagrange's auxiliary equations, and that the density of the swarm of particles can only be a function of type

$$\rho = \frac{1}{t^3} F\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t} + \frac{1}{2}gt\right)$$

5. Solve the equations

$$\begin{aligned} p + q &= \frac{z}{a} & xp + zq + y &= 0 \\ xp + yq &= nz & \frac{p}{x} + \frac{q}{y} &= \frac{1}{z} \\ x^2p + y^2q &= z^2 & (x + y)p + (y - x)q &= 0 \\ z - px - qy &= a\sqrt{x^2 + y^2 + z^2} \end{aligned}$$

where a and n are constants.

§ 51. *Special solutions of Lagrange's equation.* If u and v are not both single-valued functions of x, y, z , the equation $\phi(u, v) = 0$ may not furnish all the solutions of the partial differential equation $Pp + Qq = R$. Other solutions not included in the general integral may sometimes be found as follows.

Form the complete integral

$$v = au + b$$

where a, b are arbitrary constants, and if possible rationalise it or get rid of functions which are not single valued. If the resulting equation is

$$f(x, y, z, a, b) = 0$$

apply the rule for finding the singular integral; in other words, eliminate

a and b from the equations $f = 0, \frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0$.

Example 1. Consider the equation

$$y(z^2 - x^2 - y^2)p + (zq - y)x = 0$$

Lagrange's subsidiary equations

$$\frac{dx}{y(z^2 - x^2 - y^2)} = \frac{dy}{zx} = \frac{dz}{xy}$$

possess a first integral $z^2 - y^2 = \alpha$. Using this to modify the equations, we get

$$\frac{x dx}{\alpha - x^2} = \frac{y dy}{\sqrt{y^2 + \alpha}}$$

A second integral is now obtained in the form

$$\frac{1}{2} \log (\alpha - x^2) + \sqrt{y^2 + \alpha} = \beta$$

or

$$\frac{1}{2} \log (z^2 - x^2 - y^2) + z = \beta$$

Writing $u = z^2 - y^2$ $v = \frac{1}{2} \log (z^2 - x^2 - y^2) + z$

we form the complete integral

$$\frac{1}{2} \log (z^2 - x^2 - y^2) + z = a(z^2 - y^2) + b$$

This contains the many-valued function $\log (z^2 - x^2 - y^2)$. To get rid of this we take the exponential of both sides. Hence

$$z^2 - x^2 - y^2 = \exp [2a(z^2 - y^2) + 2(b - z)]$$

The rule for finding the singular integral now leads to the equation

$$z^2 - x^2 - y^2 = 0$$

which evidently provides us with a solution of the differential equation.

This integral is said to be *special*, because it cannot be derived directly from the differential equation like most singular solutions. It should be noticed that in this case the coefficient of p in the differential equation is zero in virtue of the relation $z^2 = x^2 + y^2$.

Example 2. Consider the equation

$$p\sqrt{z-x} + q + \sqrt{z-x} = 0$$

In this case $u = \sqrt{z-x} + y$ $v = z + x$

The complete integral

$$z + x = a[\sqrt{z-x} + y] + b$$

can be written in the form

$$(z + x - b - ay)^2 = a^2(z - x)$$

A special solution is evidently given by $z = x$, for the differential equation is satisfied. This solution is not contained in the general integral

$$F[\sqrt{z-x} + y, z+x] = 0$$

The solution may be obtained directly from the differential equation if we write this in the form

$$\varphi \equiv (p+1)^2(z-x) - q^2 = 0$$

The equations $\varphi = 0$ $\frac{\partial \varphi}{\partial p} = 0$ $\frac{\partial \varphi}{\partial q} = 0$

are satisfied by $z = x$, $q = 0$, and the equations

$$\frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z} = 0 \quad \frac{\partial \varphi}{\partial y} + q \frac{\partial \varphi}{\partial z} = 0$$

are also satisfied ; hence $z = x$ is a true singular integral.

The theory of special solutions may also be studied with the aid

of Cauchy's existence theorem,* but the analysis is beyond the scope of the present work.

The theory indicates that an integral $F(x, y, z) = 0$ may not be included in the equation $\varphi(u, v) = 0$

if either $P = 0$, or $Q = 0$, or $R = 0$ in virtue of $F = 0$, or P, Q, R are not all expressible as convergent power series of type

$$\sum_m \sum_n \sum_p A_{m,n,p} (x-c)^m (y-\alpha)^n (z-\beta)^p$$

converging for sufficient values of $|x-c|, |y-\alpha|, |z-\beta|$, where c, α, β are particular values of the variables x, y, z satisfying the equation $F(c, \alpha, \beta) = 0$ and $m+1, n+1, p+1$ are positive integers. The case in which P, Q and R are not all single-valued functions of x, y, z is included in the second alternative. It must not be concluded that these are the only ways in which special solutions can arise. If we make the substitution

$$x = x_1 + y_1 \quad y = x_1 - y_1$$

we have $p_1 = \frac{\partial z}{\partial x_1} = p + q \quad q_1 = \frac{\partial z}{\partial y_1} = p - q$

and the equation $Pp + Qq = R$ takes the form $P_1 p_1 + Q_1 q_1 = R_1$, where $P_1 + Q_1 = P, P_1 - Q_1 = Q, R_1 = R$. If now the equation $Pp + Qq = R$ possesses a special solution which makes $P = 0$, the equation $P_1 p_1 + Q_1 q_1 = R_1$ possesses a special solution which makes $P_1 + Q_1 = 0$.

The term 'special solution' has been extended so as to include solutions of type $F(x, y, z) = 0$, in which F cannot be expressed in the form $\varphi(u, v)$, although an equation of type $\varphi(u, v)$ gives some

or all of the values of z furnished by $F(x, y, z) = 0$. Thus $z - \frac{x^2}{y} = 0$

is regarded as a special solution of the equation $xp + yq = z$ because $z - \frac{x^2}{y}$ cannot be expressed as a function of $\frac{z}{x}$ and $\frac{x}{y}$. Nevertheless

the equation $\frac{z}{x} = \frac{x}{y}$ gives the same expression for z as $z - \frac{x^2}{y} = 0$.

Cf. L. L. Steimley, *American Journal* (1915). This author endeavours to find an expression of type $F(x, y, z) = \theta(x, y, z)\varphi(u, v)$ which, when equated to zero, will give both the general and special solutions.

EXAMPLES.

1. Prove that the equation

$$\begin{aligned} & [(x-y)(z^2 - 2x^2 - 2y^2) + z(x+y)]p \\ & + [(x-y)(z^2 - 2x^2 - 2y^2) - z(x+y)]q = 2(x^3 - y^3) \end{aligned}$$

* See Forsyth's *Differential Equations*, pp. 394-400. The existence of special solutions was noticed by G. Chrystal, *Trans. Roy. Soc. Edin.*, Vol. 36 (1892), p. 551, and E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, § 18.

possesses the special solution

$$z^2 = 2x^2 + 2y^2.$$

2. If a function $w(x, y, z)$ is such that the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes in virtue of the equation $w = 0$ but not identically, then $w = 0$ generally gives a special solution of the partial differential equation of which $u = \alpha, v = \beta$ are particular integrals. (Goursat.)

3. Prove that if m and n are constants and $f(x, y, z)$ an arbitrary function of x, y and z , the equation

$$m\left(p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x}\right) + n\left(q \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}\right) = f$$

possesses the general solution $\varphi[x - m \log f, y - n \log f] = 0$ and the special solution $f(x, y, z) = 0$.

§ 52. *The homogeneous linear equation.* The equation

$$Pp + Qq = R$$

is satisfied by the function z defined by the equation

$$\psi(x, y, z) = 0$$

if

$$P \frac{\partial \psi}{\partial x} + Q \frac{\partial \psi}{\partial y} + R \frac{\partial \psi}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This partial differential equation for the determination of ψ is said to be homogeneous because the same number of derivatives occur in each term.

To solve the equation by Lagrange's method, we form the subsidiary system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{d\psi}{0}$$

$$\text{Let } u(x, y, z) = a \quad v(x, y, z) = b \quad \psi = c$$

be three independent integrals of these equations; then the general integral is given by the equation

$$F(u, v, \psi) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which determines ψ as a function of u and v , and consequently as a function of x, y, z . Conversely, if $\zeta = \psi(x, y, z)$ is a solution of

$$P \frac{\partial \zeta}{\partial x} + Q \frac{\partial \zeta}{\partial y} + R \frac{\partial \zeta}{\partial z} = 0$$

we have, on substituting,

$$\lambda P = \frac{\partial(u, v)}{\partial(y, z)} \quad \lambda Q = \frac{\partial(u, v)}{\partial(z, x)} \quad \lambda R = \frac{\partial(u, v)}{\partial(x, y)} \quad \frac{\partial(\zeta, u, v)}{\partial(x, y, z)} = 0$$

This equation is satisfied identically and not in virtue of the relation $u = a, v = b, \zeta = \psi(x, y, z)$; hence ζ is a function of u and v , and so all the solutions of the differential equation (1) are given by (2).

It should be noticed that if u and v are not single-valued functions of x, y, z , the method of deriving singular integrals from a rationalised form of the complete integral

$$\psi = au + bv + c$$

leads only to relations involving x, y, z and not ψ ; hence there are no proper special integrals of the equation.

§ 53. *Simple types of soluble partial differential equations. Standard forms.* It is natural to inquire what form a partial differential equation must have in order that a complete integral may represent a family of ∞^2 planes. There are evidently two forms which the complete integral can have.

Standard I. Let the complete integral be

$$z = ax + by + c$$

where c is a constant and a and b are connected by the relation

$$\psi(a, b) = 0$$

Since $p = a, q = b$, the partial differential equation is

$$\psi(p, q) = 0$$

If with the aid of the equation $\psi(a, b) = 0$ we express a and b as functions of a single parameter α , say $a = f(\alpha), b = g(\alpha)$, the complete integral takes the form

$$z = xf(\alpha) + yg(\alpha) + c$$

To find the general integral we must make c a function of α and then find the envelope of the corresponding system of planes. Now the envelope of a family of planes depending on one parameter is a developable surface; hence the differential equation $\psi(p, q) = 0$ is the partial differential equation of a family of developable surfaces. There is evidently no singular integral.

Standard II. Let the complete integral be

$$z = ax + by + c$$

where

$$c = \psi(a, b)$$

Since $p = a, q = b$, the partial differential equation is

$$z = px + qy + \psi(p, q)$$

In this case there is a singular integral obtained by eliminating a and b from the equations

$$z = ax + by + \psi(a, b)$$

$$0 = x + \frac{\partial \psi}{\partial a}$$

$$0 = y + \frac{\partial \psi}{\partial b}$$

This singular integral evidently represents the envelope of the family of planes; it can also be obtained directly from the partial differential equation.

There are certain equations which may be solved most easily by means of special artifices.

Standard III. Equations in which the independent variables do not occur explicitly. The general form of such an equation is

$$\chi(z, p, q) = 0$$

Let us assume as a tentative solution

$$z = f(x + ay) = f(\xi)$$

where a is an arbitrary constant and $\xi = x + ay$. We then have

$$p = \frac{dz}{d\xi} \frac{\partial \xi}{\partial x} = \frac{dz}{d\xi}$$

$$q = \frac{dz}{d\xi} \frac{\partial \xi}{\partial y} = a \frac{dz}{d\xi}$$

and the substitution of these in the partial differential equation gives

$$\chi\left(z, \frac{dz}{d\xi}, a \frac{dz}{d\xi}\right) = 0$$

This is no longer a partial differential equation, as there is now only one independent variable. If we solve for $\frac{dz}{d\xi}$, we obtain an equation of the form

$$\frac{dz}{d\xi} = \varphi(z, a)$$

the solution of which is $\xi + b = \int \frac{dz}{\varphi(z, a)}$

Hence a complete integral of the partial differential equation is given by

$$x + ay + b = \int \frac{dz}{\varphi(z, a)} = F(z, a), \text{ say}$$

The general integral may be found by the ordinary method. If $F(z, a)$ is a single-valued function of z and a , there is no singular integral, but $F(z, a)$ is not generally single valued, and the singular integral may be obtained by the ordinary method after the equation has been rationalised or cleared of many-valued functions.

Sometimes an equation may be reduced to the standard form III by simple substitutions. In making a transformation of variables from x, y, z to X, Y, Z , the values of P and Q may be calculated very easily by using the fact that there must be a relation of the form

$$dz - p dx - q dy \equiv \mu(dZ - P dX - Q dY)$$

where μ is some multiplier, for the left-hand side vanishes when

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}$$

and the right-hand side when

$$P = \frac{\partial Z}{\partial X} \quad Q = \frac{\partial Z}{\partial Y}$$

Standard IV. Variables separated. When a partial differential equation can be thrown into the form

$$\varphi(x, p) = \psi(y, q)$$

a complete integral may be obtained by putting each side of the equation equal to a constant a . Solving the equations $\varphi = a$, $\psi = a$ for p and q respectively, we get

$$p = \theta_1(x, a)$$

$$q = \theta_2(y, a)$$

Substituting in the equation

$$dz = p dx + q dy$$

and integrating, we obtain the complete integral

$$z = \int \theta_1(x, a) dx + \int \theta_2(y, a) dy + b$$

wherein b is an arbitrary constant.

The preceding methods are all particular cases of the following general method due to Charpit.

Let
$$f(x, y, z, p, q) = 0$$

be the given partial differential equation; we shall endeavour to find a second relation between x, y, z, p, q , viz. :

$$\varphi(x, y, z, p, q) = 0$$

such that when p and q are expressed in terms of x, y, z with the aid of the equations $f = 0$, $\varphi = 0$, the equation

$$dz = p dx + q dy$$

becomes integrable either as it stands or after multiplication by a suitable factor. Methods of solving such an equation will be considered in the next chapter; we shall assume here that the equation can be solved and z, p, q expressed as functions of x and y .

When these expressions for z, p, q are substituted in f and φ , the equations $f = 0$, $\varphi = 0$ should be satisfied identically. The equations obtained by differentiating them with regard to x and y respectively should therefore be satisfied; hence

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial y} = 0$$

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial z} p + \frac{\partial \varphi}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \varphi}{\partial q} \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z} q + \frac{\partial \varphi}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial \varphi}{\partial q} \frac{\partial q}{\partial y} = 0$$

Solving these equations for $\frac{\partial p}{\partial y}$, $\frac{\partial q}{\partial x}$ respectively, we get

$$\frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial \varphi}{\partial y} + q \left(\frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial \varphi}{\partial z} \right) + \frac{\partial p}{\partial y} \left(\frac{\partial f}{\partial p} \frac{\partial \varphi}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial \varphi}{\partial p} \right) = 0$$

$$\frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial \varphi}{\partial x} + p \left(\frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial \varphi}{\partial z} \right) + \frac{\partial q}{\partial x} \left(\frac{\partial f}{\partial q} \frac{\partial \varphi}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial \varphi}{\partial q} \right) = 0$$

Now
$$\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$$

hence, adding the last two equations, we get

$$\left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial \varphi}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial \varphi}{\partial q} - \left(p \frac{\partial f}{\partial p} + q \frac{\partial f}{\partial q} \right) \frac{\partial \varphi}{\partial z} - \frac{\partial f}{\partial p} \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial q} \frac{\partial \varphi}{\partial y} = 0$$

This may be regarded as a linear partial differential equation for the determination of φ . To solve it by Lagrange's method, we must write down the auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{- \frac{\partial f}{\partial p}} = \frac{dy}{- \frac{\partial f}{\partial q}} = \frac{d\varphi}{0}$$

These are identical with the differential equations of the characteristics, § 49. One integral is, of course, $\varphi = \text{const.}$ If, however, another integral $F(x, y, z, p, q) = 0$ can be obtained,

$$\varphi = F(x, y, z, p, q)$$

will be a solution of the partial differential equation for φ .

In selecting an integral of the auxiliary equations choice should be made of one which involves at least one of the quantities p and q .

EXAMPLES.

1. A swarm of particles moves with constant velocity c along straight lines lying in a plane. Prove that if the curve $t = f(x, y)$ always consists of the same particles and is orthogonal to their paths, f satisfies the partial differential equation

$$\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 = \frac{1}{c^2}$$

Obtain the complete integral

$$ct = x \cos a + y \sin a + b$$

and show that the solution $ct = \sqrt{x^2 + y^2}$ is a particular case of the general integral. Prove that

$$(x - a)^2 + (y - b)^2 = c^2 t^2$$

is also a complete integral.

2. Prove that the equation

$$\frac{z - px - qy}{\sqrt{1 + p^2 + q^2}} = zf\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

can be reduced to a soluble form by the substitutions

$$X = \log(x^2 + y^2) \quad Y = \tan^{-1} \frac{y}{x} \quad Z = \frac{z}{\sqrt{x^2 + y^2}}$$

3. Solve the equations

$$\begin{aligned} px + qy &= pq & q &= px + p^2 \\ Vp + Vq &= 2x & q &= (z + px)^2 \\ (p^2 + q^2)y &= qz & p(q^2 + 1) + (b - z)q &= 0 \\ xq &= yp + xe^{x^2+y^2} \end{aligned}$$

§ 54. *The principle of duality. Contact transformations.* We can evidently obtain a relation of type

$$dz - p dx - q dy = \mu(dZ - P dX - Q dY) \quad (1)$$

which is characteristic of a *contact transformation*, by writing

$$p dx = d(px) - x dp \quad q dy = d(qy) - y dq$$

thus $dz - p dx - q dy = d(z - px - qy) + x dp + y dq$

Hence the relation (1) is satisfied by

$$\begin{aligned} \mu &= -1 & Z &= px + qy - z & X &= p \\ Y &= q & P &= x & Q &= y \end{aligned} \quad (2)$$

The transformation which is defined in this way enables us to pass from a solution of the equation

$$\varphi(x, y, z, p, q) = 0 \quad (3)$$

to a solution of the equation

$$\varphi(P, Q, PX + QY - Z, X, Y) = 0 \quad (4)$$

Thus, if $f(x, y, z) = 0$ (5)

is a solution of (3), we have

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0$$

or $\frac{\partial f}{\partial x} + X \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial y} + Y \frac{\partial f}{\partial z} = 0$ (6)

Also $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + (Z + z) \frac{\partial f}{\partial z} = 0$ (7)

Eliminating x, y, z between the equations (5), (6), (7), we obtain a relation between X, Y, Z , which is a solution of (4).

To obtain the formulae of transformation of the second derivatives, we notice that

$$\begin{aligned} dp - r dx - s dy &= dX - r dP - s dQ \\ dq - s dx - t dy &= dY - s dP - t dQ \end{aligned}$$

$$\text{Now, if } r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad t = \frac{\partial^2 z}{\partial y^2}$$

$$R = \frac{\partial^2 Z}{\partial X^2} \quad S = \frac{\partial^2 Z}{\partial X \partial Y} \quad T = \frac{\partial^2 Z}{\partial Y^2}$$

$$\text{we have } dp = r dx + s dy \quad dq = s dx + t dy$$

$$dP = R dX + S dY \quad dQ = S dX + T dY$$

Substituting in the preceding equations, we get

$$rR + sS = 1 \quad rS + sT = 0$$

$$sR + tS = 0 \quad sS + tT = 1$$

$$\therefore \frac{r}{T} = -\frac{s}{S} = \frac{t}{R} = rt - s^2$$

Another transformation is obtained by noticing that

$$dz - p dx - q dy = d(z - px) + x dp - q dy$$

Hence we may put

$$\mu = 1 \quad Z = z - px \quad X = p \quad P = -x \quad Y = y \quad Q = q$$

The partial differential equation (3) is in this case transformed into

$$\varphi(-P, Y, Z - PX, X, Q) = 0$$

If a solution of one equation is known, a solution of the other equation can be found by a process of elimination. In this case

$$dp - r dx - s dy = dX + r dP - s dY$$

$$dq - s dx - t dy = dQ + s dP - t dY$$

$$\text{Hence } R = -\frac{1}{r} \quad S = \frac{s}{r} \quad T = t - Ss = t - \frac{s^2}{r}$$

EXAMPLES.

1. Show that the transformation

$$\begin{aligned} X &= p & Y &= q & P &= x \\ Q &= y & Z &= px + qy - z & PX + QY - Z &= z \end{aligned}$$

may be interpreted geometrically as a reciprocation with regard to the paraboloid of revolution $x^2 + y^2 = 2z$.

2. Show that the characteristics of the equation $\varphi(x, y, z, p, q)$ are transformed by the substitution (2) into the characteristics of the equation

$$\varphi(P, Q, PX + QY - Z, X, Y) = 0$$

3. Reduce the equations

$$F(z - px - qy, x, y) = 0 \quad F(z - px, x, q) = 0$$

to standard forms.

4. Solve the equations

$$\begin{aligned} pq + xy &= 0 & q(z - px) &= x & pq + xy &= 1 \\ (z - px - qy)(px + qy) + pq &= 0 \end{aligned}$$

5. A family of circles is defined by a relation $\rho = f(\xi, \eta)$ between the radius ρ and the coordinates (ξ, η) of the centre.

A circle (ρ_0, ξ_0, η_0) is drawn to cut the three circles

$$(\rho, \xi, \eta) \quad \left(\rho + \frac{\partial \rho}{\partial \xi} d\xi, \xi + d\xi, \eta \right) \quad \left(\rho + \frac{\partial \rho}{\partial \eta} d\eta, \xi, \eta + d\eta \right)$$

orthogonally. Prove that

$$\xi_0 = \xi - \rho \frac{\partial \rho}{\partial \xi} \quad \eta_0 = \eta - \rho \frac{\partial \rho}{\partial \eta} \quad \rho_0 = \rho \sqrt{\left(\frac{\partial \rho}{\partial \xi} \right)^2 + \left(\frac{\partial \rho}{\partial \eta} \right)^2 - 1}$$

and deduce that the following transformation is a contact transformation:

$$\begin{aligned} X &= x - zp & Y &= y - zq & Z &= z\sqrt{p^2 + q^2 - 1} \\ P &= -\frac{p}{\sqrt{p^2 + q^2 - 1}} & Q &= \frac{-q}{\sqrt{p^2 + q^2 - 1}} \end{aligned}$$

6. Prove that if a is a constant, the transformation

$$\begin{aligned} X &= x + \frac{ap}{\sqrt{1 + p^2 + q^2}} & Y &= y + \frac{aq}{\sqrt{1 + p^2 + q^2}} \\ Z &= z - \frac{a}{\sqrt{1 + p^2 + q^2}} & P &= p & Q &= q \end{aligned}$$

is a contact transformation. What is the geometrical interpretation of the transformation?

§ 55. *Equations with three independent variables.* When there are several independent variables, there are also standard forms of equations which can be solved by simple means. For instance, a complete integral of the equation

$$F\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = 0 \quad \dots \quad (1)$$

may be obtained by writing

$$V = ax + by + cz + d \quad \dots \quad (2)$$

where

$$F(a, b, c) = 0 \quad \dots \quad (3)$$

To obtain the general integral we make d a function of a, b, c and eliminate a, b, c from the equations

$$\begin{aligned} 0 &= x + z \frac{\partial c}{\partial a} + \frac{\partial d}{\partial a} + \frac{\partial d}{\partial c} \frac{\partial c}{\partial a} \\ 0 &= y + z \frac{\partial c}{\partial b} + \frac{\partial d}{\partial b} + \frac{\partial d}{\partial c} \frac{\partial c}{\partial b} \\ 0 &= \frac{\partial F}{\partial a} + \frac{\partial F}{\partial c} \frac{\partial c}{\partial a} \\ 0 &= \frac{\partial F}{\partial b} + \frac{\partial F}{\partial c} \frac{\partial c}{\partial b} \end{aligned}$$

and the equations already given.

When F is a homogeneous function of its arguments, $f(V)$ is a solution of the partial differential equation when V is a solution. In this case solutions of the partial differential equation may be obtained in a number of ways, which may be described as follows:

1°. Let a, b, c be functions of V which satisfy the equation (3); then if V is defined by the equation (2), it satisfies equation (1). The quantity d may be an arbitrary function of V .

2°. Let $F(\xi, \eta, \zeta) = 0$ be interpreted as the condition that the line $\xi x + \eta y + \zeta = 0$ may touch a curve given by the parametric equations

$$x = f(\alpha) \quad y = g(\alpha)$$

then, since the tangent at the point with parameter α is

$$xg'(\alpha) - yf'(\alpha) = f(\alpha)g'(\alpha) - f'(\alpha)g(\alpha)$$

the relation $F(\xi, \eta, \zeta) = 0$ is satisfied when

$$\xi = g'(\alpha) \quad \eta = -f'(\alpha) \quad \zeta = f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)$$

$$\text{Now write} \quad x = zf(\alpha) + u \quad y = zg(\alpha) + v$$

then

$$\begin{aligned} dx &= dzf(\alpha) + zf'(\alpha) d\alpha + du \\ dy &= dzg(\alpha) + zg'(\alpha) d\alpha + dv \end{aligned}$$

and so

$$g'(\alpha) dx - f'(\alpha) dy + dz[f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)] = g'(\alpha) du - f'(\alpha) dv$$

If α is a function of u and v , we may write

$$g'(\alpha) du - f'(\alpha) dv = \mu d\theta$$

and then

$$\mu \frac{\partial \theta}{\partial x} = g'(\alpha) = \xi$$

$$\mu \frac{\partial \theta}{\partial y} = -f'(\alpha) = \eta \quad \mu \frac{\partial \theta}{\partial z} = f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha) = \zeta$$

consequently

$$F\left(\mu \frac{\partial \theta}{\partial x}, \mu \frac{\partial \theta}{\partial y}, \mu \frac{\partial \theta}{\partial z}\right) = 0$$

We may conclude from this, that if $\Phi(\theta)$ is an arbitrary function, the equation (1) is satisfied by $V = \Phi(\theta)$. We thus have the following result:

If $F(\xi, \eta, \zeta) = 0$ is the condition that the line $\xi x + \eta y + \zeta = 0$ may touch the curve $x = f(\alpha), y = g(\alpha)$, a solution of the differential equation (1) is obtained by making α an arbitrary function of u and v , solving the equations

$$x = zf(\alpha) + u \quad y = zg(\alpha) + v$$

for u and v and determining θ from the equation

$$g'(\alpha) du - f'(\alpha) dv = \mu d\theta$$

The partial differential equation is then satisfied by $V = \Phi(\theta)$.

EXAMPLES.

1. Prove that the equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = 0$$

may be solved by writing $V = \Phi(\theta)$, where

$$\begin{aligned} x &= iz \cos \alpha + u & y &= iz \sin \alpha + v & \alpha &= F(u, v) \\ \cos \alpha \, du + \sin \alpha \, dv &= \mu \, d\theta \end{aligned}$$

2. Deduce from the preceding result that if
- θ
- is defined by the equation

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0$$

where ξ, η, ζ are arbitrary functions, the expression $V = \Phi(\theta)$ satisfies the partial differential equation.

3. Obtain a general solution of the equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \left(\frac{\partial V}{\partial t}\right)^2$$

CHAPTER VII

TOTAL DIFFERENTIAL EQUATIONS

§ 56. *The condition of integrability.* If a particle is moving on a surface whose equation is $F(x, y, z) = 0$, the component velocities $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ of the particle are subject to the relation

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0 \quad . \quad . \quad . \quad (1)$$

which expresses that the direction in which the particle is moving is at right angles to the normal to the surface. This equation may be written in the abbreviated form

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0 \quad . \quad . \quad . \quad (2)$$

Now let us take the more general case in which the direction of motion of the particle is subject to the condition

$$P dx + Q dy + R dz = 0 \quad . \quad . \quad . \quad (3)$$

where P, Q, R are functions of x, y, z but not of t . It is natural to ask if this condition implies that the particle always lies on a surface of the family $F(x, y, z) = \text{const.}$ whatever may be the initial position of the particle and its initial direction of motion. The answer is, that this is not necessarily the case, for if we assume that equation (3) is simply a multiple of equation (2), so that

$$P = \mu \frac{\partial F}{\partial x} \quad Q = \mu \frac{\partial F}{\partial y} \quad R = \mu \frac{\partial F}{\partial z} \quad . \quad . \quad (4)$$

where μ is some multiplier, we find that

$$\left. \begin{aligned} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= \frac{\partial \mu}{\partial y} \frac{\partial F}{\partial z} - \frac{\partial \mu}{\partial z} \frac{\partial F}{\partial y} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= \frac{\partial \mu}{\partial z} \frac{\partial F}{\partial x} - \frac{\partial \mu}{\partial x} \frac{\partial F}{\partial z} \\ \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} &= \frac{\partial \mu}{\partial x} \frac{\partial F}{\partial y} - \frac{\partial \mu}{\partial y} \frac{\partial F}{\partial x} \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

Hence the quantities P, Q, R must satisfy the condition

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \quad (6)$$

Denoting the three quantities on the left-hand side of (5) by ξ, η, ζ respectively, the condition takes the form

$$P\xi + Q\eta + R\zeta = 0$$

This condition is necessary, and it will be shown presently that it is also sufficient; in other words, when the condition (6) is satisfied, the differential form (3) can be thrown into the form (2).

We shall find it convenient in the following analysis to denote the differential form on the left-hand side of (3) by the symbol Λ .

EXAMPLES.

1. Prove that the condition of integrability is satisfied in the following cases:

$$(y - z) dx + (z - x) dy + (x - y) dz = 0$$

$$e^x dx + (e^x - e^y) dy + (e^y - e^z) dz = 0$$

$$(y^2 + yz) dx + (xz + z^2) dy + (y^2 - xy) dz = 0$$

2. Prove that if $P dx + Q dy + R dz + S dt$ can be reduced to the form $U du$, the equations

$$S\xi + Q\gamma - R\beta = 0 \quad S\zeta + P\beta - Q\alpha = 0$$

$$S\eta + R\alpha - P\gamma = 0 \quad P\xi + Q\eta + R\zeta = 0$$

are satisfied where

$$\begin{aligned} \xi &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & \eta &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} & \zeta &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ \alpha &= \frac{\partial S}{\partial x} - \frac{\partial P}{\partial t} & \beta &= \frac{\partial S}{\partial y} - \frac{\partial Q}{\partial t} & \gamma &= \frac{\partial S}{\partial z} - \frac{\partial R}{\partial t} \end{aligned}$$

3. Prove that if $P dx + Q dy + R dz + S dt$ can be reduced to the form $dw + v du$, the condition

$$\xi\alpha + \eta\beta + \zeta\gamma = 0$$

is satisfied.

§ 57. *Transformation of variables.* If the variables are changed from x, y, z to u, v, w by a transformation of type

$$x = f(u, v, w) \quad y = g(u, v, w) \quad z = h(u, v, w)$$

so that
$$dx = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial w} dw$$

and similar equations for dy, dz , the differential form Λ is transformed into

$$\Lambda = U du + V dv + W dw$$

where U, V, W are certain functions of u, v, w depending on the forms of P, Q, R, f, g, h . Our object is to find the relation between U, V, W which corresponds to the relation (6) between P, Q, R .

Now, in transforming a relation such as (6), the following symbolical method can be used with advantage. This method is of wide application.

Let (dx, dy, dz) , $(\delta x, \delta y, \delta z)$, $(\partial x, \partial y, \partial z)$ be three independent sets of increments of the variables (x, y, z) ; (du, dy, dz) , $(\delta u, \delta y, \delta z)$, $(\partial u, \partial y, \partial z)$ the corresponding sets of increments of the variables u, v, w . Then

$$P dx + Q dy + R dz = U du + V dv + W dw$$

$$P \delta x + Q \delta y + R \delta z = U \delta u + V \delta v + W \delta w$$

$$P \partial x + Q \partial y + R \partial z = U \partial u + V \partial v + W \partial w$$

If $A dx + B dy + C dz$, $L dx + M dy + N dz$ are any two differential forms, the determinant

$$\begin{vmatrix} A dx + B dy + C dz & L dx + M dy + N dz \\ A \delta x + B \delta y + C \delta z & L \delta x + M \delta y + N \delta z \end{vmatrix}$$

will be called their *symbolical product*; it is equal to

$$(BN - CM) d(y, z) + (CL - AN) d(z, x) + (AM - BL) d(x, y) \quad (7)$$

where the symbols $d(y, z)$, $d(z, x)$, $d(x, y)$ are used to denote the determinants

$$\begin{vmatrix} dy & dz \\ \delta y & \delta z \end{vmatrix} \quad \begin{vmatrix} dz & dx \\ \delta z & \delta x \end{vmatrix} \quad \begin{vmatrix} dx & dy \\ \delta x & \delta y \end{vmatrix}$$

respectively. Notice that $d(y, z) = -d(z, x)$, $d(x, x) = 0$. The form (7) may, moreover, be derived from $A dx + B dy + C dz$ and $L dx + M dy + N dz$ by actual multiplication if we multiply the coefficients A, B, C, L, M, N by the ordinary rules of multiplication and multiply the increments dx, dy , etc., according to the rules

$$\begin{aligned} dy \times dz &= d(y, z) & dz \times dy &= d(z, y) = -d(y, z) \\ dx \times dx &= d(x, x) = 0 \end{aligned}$$

Similarly, we can define the symbolical product of three forms $A dx + B dy + C dz$, $L dx + M dy + N dz$, $P dx + Q dy + R dz$ as the determinant

$$\begin{vmatrix} A dx + B dy + C dz & L dx + M dy + N dz & P dx + Q dy + R dz \\ A \delta x + B \delta y + C \delta z & L \delta x + M \delta y + N \delta z & P \delta x + Q \delta y + R \delta z \\ A \partial x + B \partial y + C \partial z & L \partial x + M \partial y + N \partial z & P \partial x + Q \partial y + R \partial z \end{vmatrix}$$

It evidently has the value $\Delta d(x, y, z)$, where the symbols $\Delta, d(x, y, z)$ denote the determinants

$$\begin{vmatrix} A & B & C \\ L & M & N \\ P & Q & R \end{vmatrix} \quad \begin{vmatrix} dx & dy & dz \\ \delta x & \delta y & \delta z \\ \partial x & \partial y & \partial z \end{vmatrix}$$

respectively. The quantity $\Delta d(x, y, z)$ can be obtained by direct multiplication of

$$A dx + B dy + C dz \quad L dx + M dy + N dz \quad P dx + Q dy + R dz$$

or of (7), and $P dx + Q dy + R dz$ if we multiply the increments dx, dy, dz according to the rules

$$\begin{aligned} dx \times dy \times dz &= d(x, y, z) & dx \times d(y, z) &= d(x, y, z) & d(x, x, z) &= 0 \\ d(x, y, z) &= -d(y, x, z) = d(y, z, x) = -d(z, y, x) \\ d(x, z, z) &= 0 & d(x, x, x) &= 0 & d(x, y, x) &= 0 \end{aligned}$$

It is now seen that if

$$F = BN - CM \quad G = CL - AN \quad H = AM - BL$$

the symbolical product of

$$P dx + Q dy + R dz \quad \text{and} \quad F d(y, z) + G d(z, x) + H d(x, y)$$

is $(PF + QG + RH) d(x, y, z)$

This holds for arbitrary values of F, G, H .

Now let symbolical differentiation be defined as the operation whereby $P dx + Q dy + R dz$ is replaced by

$$d(P, x) + d(Q, y) + d(R, z)$$

and $F d(y, z) + G d(z, x) + H d(x, y)$

by $d(F, y, z) + d(G, z, x) + d(H, x, y)$

If we form the symbolical product of du and dv , replacing these quantities by

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad \text{and} \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

respectively, we find that

$$d(u, v) = \frac{\partial(u, v)}{\partial(x, y)} d(x, y) + \frac{\partial(u, v)}{\partial(y, z)} d(y, z) + \frac{\partial(u, v)}{\partial(z, x)} d(z, x)$$

hence

$$d(P, x) = \frac{\partial P}{\partial z} d(z, x) - \frac{\partial P}{\partial y} d(x, y)$$

$$d(Q, y) = \frac{\partial Q}{\partial x} d(x, y) - \frac{\partial Q}{\partial z} d(y, z)$$

$$d(R, z) = \frac{\partial R}{\partial y} d(y, z) - \frac{\partial R}{\partial x} d(z, x)$$

The symbolical derivative of $P dx + Q dy + R dz$ is thus

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) d(y, z) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) d(z, x) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y)$$

Notice that if

$$U du = A dx + B dy + C dz = U \frac{\partial u}{\partial x} dx + U \frac{\partial u}{\partial y} dy + U \frac{\partial u}{\partial z} dz$$

its symbolical derivative is the same whether it is calculated from the expression on the left-hand side or its expression on the right-

hand side, for in one case the derivative is $d(U, u)$, and in the other case it is

$$\left[\frac{\partial}{\partial y} \left(U \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial z} \left(U \frac{\partial u}{\partial y} \right) \right] d(y, z) + \left[\frac{\partial}{\partial z} \left(U \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left(U \frac{\partial u}{\partial z} \right) \right] d(z, x) \\ + \left[\frac{\partial}{\partial x} \left(U \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(U \frac{\partial u}{\partial x} \right) \right] d(x, y)$$

or
$$\frac{\partial(U, u)}{\partial(y, z)} d(y, z) + \frac{\partial(U, u)}{\partial(z, x)} d(z, x) + \frac{\partial(U, u)}{\partial(x, y)} d(x, y)$$

which is equal to $d(U, u)$. It should be remarked that if $d(U, u)$ is identically zero the three Jacobians in the preceding expression are zero, and so it follows that there is a relation between U and u which does not contain x, y and z . The vanishing of the first Jacobian indicates in fact that either u is a constant or that there is a relation of the form $U = f(u, x)$.

The other Jacobians then take the forms

$$- \frac{\partial f}{\partial x} \frac{\partial u}{\partial z} \quad \frac{\partial f}{\partial x} \frac{\partial u}{\partial y}$$

respectively, and indicate that either f is independent of x or that u is a function of x only. In both cases U is a function of u only.

If we form the symbolical product of du, dv, dw , replacing these quantities by

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz \\ \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

respectively, we find that

$$d(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} d(x, y, z)$$

In particular,

$$d(F, y, z) = \frac{\partial F}{\partial x} d(x, y, z) \quad d(F, z, x) = \frac{\partial F}{\partial y} d(x, y, z)$$

$$d(F, x, y) = \frac{\partial F}{\partial z} d(x, y, z)$$

hence the symbolical derivative of $F d(y, z) + G d(z, x) + H d(x, y)$ is

$$\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \right) d(x, y, z)$$

In particular, if we find the symbolical derivative of the expression

$$w d(u, v) = w \frac{\partial(u, v)}{\partial(y, z)} d(y, z) + w \frac{\partial(u, v)}{\partial(z, x)} d(z, x) + w \frac{\partial(u, v)}{\partial(x, y)} d(x, y)$$

in two ways—first, by using the expression on the left-hand side,

and secondly, by using the expression on the right-hand side—we obtain in one case $d(w, u, v)$, and in the other case

$$\left\{ \frac{\partial}{\partial x} \left[w \frac{\partial(u, v)}{\partial(y, z)} \right] + \frac{\partial}{\partial y} \left[w \frac{\partial(u, v)}{\partial(z, x)} \right] + \frac{\partial}{\partial z} \left[w \frac{\partial(u, v)}{\partial(x, y)} \right] \right\} d(x, y, z)$$

or
$$\frac{\partial(w, u, v)}{\partial(x, y, z)} d(x, y, z)$$

which is equal to $d(w, u, v)$.

We come then to the following conclusion :

In performing the operations of symbolical multiplication and differentiation with expressions of type

$$P dx + Q dy + R dz \quad F d(y, z) + G d(z, x) + H d(x, y)$$

the results are the same if we replace these expressions by their equivalents when the variables are transformed to u, v, w , provided the increments du, dv, dw are multiplied according to the same rules as dx, dy, dz .

The foregoing analysis may be extended without difficulty to the case in which there are n independent variables ; thus, when $n = 4$, the symbolical product of

$$\begin{aligned} A_1 dx + B_1 dy + C_1 dz + D_1 dt \quad \text{and} \quad A_2 dx + B_2 dy + C_2 dz + D_2 dt \\ \text{is} \quad (B_1 C_2 - B_2 C_1) d(y, z) + (C_1 A_2 - C_2 A_1) d(z, x) \\ + (A_1 B_2 - A_2 B_1) d(x, y) + (A_1 D_2 - A_2 D_1) d(x, t) \\ + (B_1 D_2 - B_2 D_1) d(y, t) + (C_1 D_2 - C_2 D_1) d(z, t) \end{aligned}$$

and the symbolical derivative of $P dx + Q dy + R dz + S dt$ is

$$\begin{aligned} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) d(y, z) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) d(z, x) \\ + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y) + \left(\frac{\partial S}{\partial x} - \frac{\partial P}{\partial t} \right) d(x, t) \\ + \left(\frac{\partial S}{\partial y} - \frac{\partial Q}{\partial t} \right) d(y, t) + \left(\frac{\partial S}{\partial z} - \frac{\partial R}{\partial t} \right) d(z, t) \end{aligned}$$

The symbolical product of

$$F d(y, z) + G d(z, x) + H d(x, y) + L d(x, t) + M d(y, t) + N d(z, t)$$

and

$$P dx + Q dy + R dz + S dt$$

$$\begin{aligned} \text{is} \quad (FS + QN - RM) d(y, z, t) + (GS + RL - PN) d(z, x, t) \\ + (HS + PM - QL) d(x, y, t) + (PF + QG + RH) d(x, y, z) \end{aligned}$$

while the symbolical derivative of the first expression is

$$\begin{aligned} \left(\frac{\partial F}{\partial t} + \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) d(y, z, t) + \left(\frac{\partial G}{\partial t} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \right) d(z, x, t) \\ + \left(\frac{\partial H}{\partial t} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) d(x, y, t) + \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial R}{\partial z} \right) d(x, y, z) \end{aligned}$$

The symbolical product of

$$A d(y, z, t) + B d(z, x, t) + C d(x, y, t) - D d(x, y, z)$$

and

$$P dx + Q dy + R dz + S dt$$

is

$$(AP + BQ + CR + DS) d(x, y, z, t)$$

while the symbolical derivative of the first expression is

$$\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} + \frac{\partial D}{\partial t} \right) d(x, y, z, t)$$

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EXAMPLES.

1. If in a transformation of variables from (x, y, z, t) to (x', y', z', t) , $\Omega \equiv P dx + Q dy + R dz + S dt$ is transformed into

$$P' dx' + Q' dy' + R' dz' + S' dt'$$

prove that the forms of the following expressions are unaltered :

$$\xi d(y, z) + \eta d(z, x) + \zeta d(x, y) + \alpha d(x, t) + \beta d(y, t) + \gamma d(z, t)$$

$$A d(y, z, t) + B d(z, x, t) + C d(x, y, t) + D d(x, y, z)$$

$$(\xi\alpha + \eta\beta + \zeta\gamma) d(x, y, z, t)$$

where $\xi = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \quad \eta = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \quad \zeta = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$\alpha = \frac{\partial S}{\partial x} - \frac{\partial P}{\partial t} \quad \beta = \frac{\partial S}{\partial y} - \frac{\partial Q}{\partial t} \quad \gamma = \frac{\partial S}{\partial z} - \frac{\partial R}{\partial t}$$

$$A = S\xi + Q\gamma - R\beta \quad C = S\zeta + P\beta - Q\alpha$$

$$B = S\eta + R\alpha - P\gamma \quad D = P\xi + Q\eta + R\zeta$$

Hence show that if the reduced form for Ω is $M du + N dv$, u and v satisfy the partial differential equation

$$A \frac{\partial V}{\partial x} + B \frac{\partial V}{\partial y} + C \frac{\partial V}{\partial z} - D \frac{\partial V}{\partial t} = 0$$

2. Prove that with the preceding notation,

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} - \frac{\partial D}{\partial t} = \xi\alpha + \eta\beta + \zeta\gamma$$

3. Prove that when u, v and w are functions of x, y, z and t ,

$$\begin{aligned} d(u, v) &= \frac{\partial(u, v)}{\partial(y, z)} d(y, z) + \frac{\partial(u, v)}{\partial(z, x)} d(z, x) + \frac{\partial(u, v)}{\partial(x, y)} d(x, y) \\ &\quad + \frac{\partial(u, v)}{\partial(x, t)} d(x, t) + \frac{\partial(u, v)}{\partial(y, t)} d(y, t) + \frac{\partial(u, v)}{\partial(z, t)} d(z, t) \\ d(u, v, w) &= \frac{\partial(u, v, w)}{\partial(y, z, t)} d(y, z, t) + \frac{\partial(u, v, w)}{\partial(z, x, t)} d(z, x, t) \\ &\quad + \frac{\partial(u, v, w)}{\partial(x, y, t)} d(x, y, t) + \frac{\partial(u, v, w)}{\partial(x, y, z)} d(x, y, z) \end{aligned}$$

4. Prove that if $d(u, v, w)$ is identically zero there is a relation between u, v , and w , and that if $d(u, v)$ is identically zero there is a relation between u and v .

5. If P, Q and R are continuous functions of x, y and z , the expression

$$P d(y, z) + Q d(z, x) + R d(x, y)$$

can be reduced to the form $W d(u, v)$. When can the expression be reduced to the form $d(u, v)$?

6. If $\xi, \eta, \zeta, \alpha, \beta, \gamma$ are continuous functions of x, y, z, t , the expression

$$\xi d(y, z) + \eta d(z, x) + \zeta d(x, y) + \alpha d(x, t) + \beta d(y, t) + \gamma d(z, t)$$

can be reduced to the form

$$W d(u, v) + P d(u, t) + Q d(v, t) + R d(w, t)$$

Hence show that if $\xi\alpha + \eta\beta + \zeta\gamma = 0$, the expression can be reduced to one of the two forms

$$\begin{aligned} P d(u, t) + Q d(v, t) + R d(w, t) \\ W d(u, v) + P d(u, t) + Q d(v, t) \end{aligned}$$

7. The necessary and sufficient conditions that the expression

$$\xi d(y, z) + \eta d(z, x) + \zeta d(x, y) + \alpha d(x, t) + \beta d(y, t) + \gamma d(z, t)$$

can be reduced to the form $Z d(X, Y)$ are that the following equations should be satisfied :

$$\begin{aligned} \alpha s + q\zeta - r\eta &= 0 & \gamma s + p\eta - q\xi &= 0 \\ \beta s + r\xi - p\zeta &= 0 & \alpha p + \beta q + \gamma r &= 0 \\ \xi\alpha + \eta\beta + \zeta\gamma &= 0 \end{aligned}$$

where

$$\begin{aligned} s &= \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} & p &= \frac{\partial\beta}{\partial z} - \frac{\partial\gamma}{\partial y} - \frac{\partial\xi}{\partial t} \\ q &= \frac{\partial\gamma}{\partial x} - \frac{\partial\alpha}{\partial z} - \frac{\partial\eta}{\partial t} & r &= \frac{\partial\alpha}{\partial y} - \frac{\partial\beta}{\partial z} - \frac{\partial\zeta}{\partial t} \end{aligned}$$

(The results of § 59 are needed in this example.)

8. In the last example, if $p = q = r = s = 0$ and $\xi\alpha + \eta\beta + \zeta\gamma = 0$, the appropriate canonical form is $d(X, Y)$. (C. Méray, A. C. Dixon.)

§ 58. *Invariance of the condition of integrability.* If we start with the equation

$$P dx + Q dy + R dz = U du + V dv + W dw$$

we deduce by symbolical differentiation that

$$\begin{aligned} & \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) d(y, z) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) d(z, x) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y) \\ &= \left(\frac{\partial W}{\partial v} - \frac{\partial V}{\partial w} \right) d(v, w) + \left(\frac{\partial U}{\partial w} - \frac{\partial W}{\partial u} \right) d(w, u) + \left(\frac{\partial V}{\partial u} - \frac{\partial U}{\partial v} \right) d(u, v) \end{aligned}$$

Multiplying this equation symbolically by the preceding one, we obtain

$$\begin{aligned} & \left[P \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + Q \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] d(x, y, z) \\ &= \left[U \left(\frac{\partial W}{\partial v} - \frac{\partial V}{\partial w} \right) + V \left(\frac{\partial U}{\partial w} - \frac{\partial W}{\partial u} \right) + W \left(\frac{\partial V}{\partial u} - \frac{\partial U}{\partial v} \right) \right] d(u, v, w) \end{aligned}$$

Hence, if the condition of integrability is satisfied in the case of the equation

$$P dx + Q dy + R dz = 0$$

it is also satisfied in the case of the transformed equation

$$U du + V dv + W dw = 0$$

If we use I and J to denote the quantities within square brackets, we find from the relation

$$d(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} d(x, y, z)$$

that

$$I = J \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Hence I is transformed into a similar expression multiplied by the Jacobian of the transformation. I is consequently an invariant.

§ 59. *First method of solving the equation $\Lambda = 0$.* If by a change of variables Λ can be reduced to the form $U du + V dv$, the condition of integrability becomes simply

$$V \frac{\partial U}{\partial w} - U \frac{\partial V}{\partial w} = 0$$

or

$$\frac{\partial}{\partial w} \left(\frac{U}{V} \right) = 0$$

Hence $\frac{U}{V}$ depends only on u and v . This means that $U du + V dv$ can be made an exact differential dF by multiplying it by a suitable integrating factor. If now we write

$$\Lambda = E dF$$

and differentiate symbolically, using the notation of § 56, we find that

$$\xi d(y, z) + \eta d(z, x) + \zeta d(x, y) = d(E, F).$$

If ξ , η and ζ are zero, we have $d(E, F) = 0$, and this implies that E is a function of F ; hence in this case Λ is an exact differential. Conversely, if Λ is an exact differential, ξ , η and ζ are zero.

To show that Λ can be reduced* to the form $U du + V dv$, we proceed as follows:

Integrate the equation $P dx + Q dy = 0$ on the supposition that z is a constant. We shall suppose that this can be done with the aid of an integrating factor, so that

$$P = \lambda \frac{\partial F}{\partial x} \quad Q = \lambda \frac{\partial F}{\partial y}$$

The integral thus obtained may be written in the form

$$F(x, y, z) = C$$

where C must be regarded as a function of z . When x , y and z vary, we have

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

consequently, we may write

$$\begin{aligned} \Lambda = P dx + Q dy + R dz &= \lambda \left(\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \right) + R dz \\ &= \lambda dF + \left(R - \lambda \frac{\partial F}{\partial z} \right) dz \end{aligned}$$

Putting $u = F$, $U = \lambda$, $v = z$, $V = R - \lambda \frac{\partial F}{\partial z}$, our object is accomplished.

This method can evidently be used to integrate the equation $\Lambda = 0$ in certain cases.

To illustrate the method, let us consider the equation

$$2xz dx + 2yz dy + (z^2 - x^2 - y^2) dz = 0$$

The equation $2xz dx + 2yz dy = 0$

is satisfied by $F \equiv x^2 + y^2 = C$ or by $z = 0$

Using the first solution, we find that the differential equation may be written in the form

$$z dF + (z^2 - F) dz = 0$$

This becomes a homogeneous equation of the first type when we write s^2 instead of F . The natural substitution is therefore $F = z^2 t$; the equation then takes the form

$$(1 + t) dz + z dt = 0$$

or

$$z(1 + t) = c$$

* It is tacitly assumed here that an ordinary differential equation of the first order possesses a solution, a proposition which we have not yet proved. It will be shown, however, in Chapter IX, that if certain reasonable conditions are satisfied, a solution undoubtedly exists.

Thus
$$z \left(1 + \frac{x^2 + y^2}{z^2} \right) = c$$

or
$$x^2 + y^2 + z^2 = cz$$

The particular solution $z = 0$ is obtained by putting $c = \infty$.

When a total differential equation

$$P dx + Q dy + R dz = 0$$

is reduced to the form $dF(x, y, z) = 0$

it does not follow that all possible solutions are included in the formula

$$F(x, y, z) = c$$

So much depends, in fact, on the nature of the function F .

Let us suppose, for instance, that

$$F(x, y, z) \equiv x + \sqrt{1 - yz} = c$$

The corresponding total differential equation is

$$dx - \frac{y dz + z dy}{2\sqrt{1 - yz}} = 0$$

and when it is written in the form

$$2\sqrt{1 - yz} dx - y dz - z dy = 0$$

it is seen that it possesses the *special solution*

$$yz = 1$$

which is not given by the general integral $x + \sqrt{1 - yz} = c$.

If we rationalise this last equation, we obtain

$$1 - yz = (c - x)^2$$

and it is seen that $1 - yz = 0$ is an envelope of the singular surfaces.

Thus $yz = 1$ is in this case a *singular solution*.

In the case of the equation

$$(y + y^2x - yz)dx + (x + x^2y - xz)dy - dz = 0$$

the general integral is $xy + \log(xy - z) = c$

but there is a *special integral* $z = xy$. If we write the general integral in the form

$$xy - z = ae^{-xy}$$

where a is a new constant, the "special integral" can be derived from the general integral by putting $a = 0$. The integral $z - xy = 0$ is consequently not a singular solution. For further information on the subject of singular solutions, the reader is referred to papers by Guldberg, *Comptes Rendus*, Dec. 26 (1898), and E. O. Lovett, *Ibid.* t. 129 (1899).

EXAMPLES.

1. Solve the equation

$$y dx - x dy + (x^2 - y^2) dz = 0$$

and discuss the nature of the solutions

$$x - y = 0 \qquad x + y = 0$$

2. Solve the equations

$$(ny - mx) dx + (lz - nx) dy + (mx - ly) dz = 0$$

$$x(y^2 + z^2) dx - y(z^2 + x^2) dy - z(x^2 + y^2) dz = 0$$

$$y(y + z) dx + z(x + z) dy + y(y - x) dz = 0$$

$$(y^2 + yz + z^2) dx + (z^2 + zx + x^2) dy + (x^2 + xy + y^2) dz = 0$$

$$z dx - x dy + (xz + x \log x) dz = 0$$

$$(y + a)^2 dx + z dy - (y + a) dz = 0$$

$$(2xz + z^2) dx + 2yz dy - 2(x^2 + y^2 + a^2) dz = 0$$

§ 60. *Canonical form for Λ . Second method of solution.* We shall now show that a change of variables can be made which will reduce Λ to the canonical form *

$$\Lambda = dw + u dv$$

Let us suppose that this has been accomplished; then the form $\Lambda - dw$ can be made an exact differential dv by dividing it by a suitable factor u , and consequently its invariant must vanish. This condition gives the partial differential equation

$$\xi \left(P - \frac{\partial w}{\partial x} \right) + \eta \left(Q - \frac{\partial w}{\partial y} \right) + \zeta \left(R - \frac{\partial w}{\partial z} \right) = 0$$

for the determination of w . Treating this by Lagrange's method, we have the auxiliary equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{dw}{\xi P + \eta Q + \zeta R}$$

If one integral $\alpha(x, y, z) = a$ of these equations can be found, we can proceed as follows: Write $v = \alpha$ and regard x, y, z as connected by the relation $\alpha(x, y, z) = a$; then $dv = 0$ and Λ reduces to an exact differential of a function w . Now, when z is eliminated with the aid of the relation $\alpha(x, y, z) = a$, the expression Λ can be thrown into the form

$$\Lambda = dG(x, y, a)$$

Let us substitute $a = \alpha(x, y, z)$ and write $G(x, y, \alpha) = w$; the function u may then be found by writing

$$P - \frac{\partial w}{\partial x} = u \frac{\partial v}{\partial x}$$

When $\alpha(x, y, z) = a$, $\beta(x, y, z) = b$, two independent solutions of the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

* This is a particular case of a general theorem concerning the reduction of

$$X_1 dx_1 + X_2 dx_2 + \dots + X_n dx_n$$

to a canonical form. See Forsyth's *Differential Equations*, p. 327; *Theory of Differential Equations*, Vol. I. The present method is sometimes called Bertrand's method.

are known, we may write

$$\xi = \gamma \frac{\partial(\alpha, \beta)}{\partial(y, z)} \quad \eta = \gamma \frac{\partial(\alpha, \beta)}{\partial(z, x)} \quad \zeta = \gamma \frac{\partial(\alpha, \beta)}{\partial(x, y)}$$

Now
$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0$$

consequently
$$\frac{\partial(\gamma, \alpha, \beta)}{\partial(x, y, z)} = 0$$

and so γ is a function of α and β . Let us write $\gamma = f(\alpha, \beta)$.

$$v = \alpha \overline{\text{MICHVBD}} u = - \int f(\alpha, \beta) d\beta$$

then
$$\frac{\partial(u, v)}{\partial(\alpha, \beta)} = f(\alpha, \beta) = \gamma$$

and so
$$\xi = \frac{\partial(u, v)}{\partial(\alpha, \beta)} \frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{\partial(u, v)}{\partial(y, z)}$$

$$\eta = \frac{\partial(u, v)}{\partial(z, x)} \quad \zeta = \frac{\partial(u, v)}{\partial(x, y)}$$

The expression $\Lambda - u dv$ is now seen to be an exact differential, dw , for

$$\frac{\partial}{\partial y} \left(R - u \frac{\partial v}{\partial z} \right) - \frac{\partial}{\partial z} \left(Q - u \frac{\partial v}{\partial y} \right) = \xi - \frac{\partial(u, v)}{\partial(y, z)} = 0$$

and similar equations.

It is easily seen that if $w = g(x, y, z)$ is one possible value of w , another possible value of w is

$$w = \Phi(u, v) + g(x, y, z)$$

where Φ is an arbitrary function. For if $\Lambda = dg + u dv$, we also have

$$\begin{aligned} \Lambda &= d(g + \Phi) + u dv - \frac{\partial \Phi}{\partial u} du - \frac{\partial \Phi}{\partial v} dv \\ &= dw + U dV \end{aligned}$$

When the expression Λ has been thrown into the form

$$dw + u dv = 0$$

there are clearly three ways in which the equation $\Lambda = 0$ may be satisfied:

1°. $v = \text{const.}$ $w = \text{const.}$

2°. $w = \text{const.}$ $u = 0$

3°. $\psi(v, w) = 0$ $u \frac{\partial \psi}{\partial w} = \frac{\partial \psi}{\partial v}$

When the invariant of Λ is zero, either of the preceding methods may be used to find the solution, which is clearly of the form

$$H(u, v) = \text{const.}$$

for the vanishing of the invariant implies that u is a function of v and w .

EXAMPLES.

1. Solve the equation

$$\Lambda \equiv yz \, dx - zx \, dy - y^2 \, dz$$

The auxiliary equations $\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$ are in this case

$$\frac{dx}{x-2y} = \frac{dy}{y} = \frac{dz}{-2z}$$

and two independent solutions are

$$\alpha \equiv y^2 z = a \qquad \beta \equiv y^2 e^{\frac{z}{y}} = b$$

$$\text{Now} \qquad \frac{\partial(\alpha, \beta)}{\partial(z, x)} = y^2 e^{\frac{z}{y}} = 2\beta\eta$$

therefore $f(\alpha, \beta) = \frac{1}{\beta}$, and so $u = - \int f(\alpha, \beta) \, d\beta = - \log \beta$

We now have

$$\begin{aligned} dw &= \Lambda - u \, d\alpha \\ &= yz \, dx - zx \, dy - y^2 \, dz + \left(\frac{x}{y} + 2 \log y \right) (2yz \, dy + y^2 \, dz) \\ &= d[xyz + zy^2(2 \log y - 1)] = d[\alpha \log \beta - \alpha] \end{aligned}$$

$$\text{Thus} \qquad \Lambda = d[\alpha \log \beta - \alpha] - \log \beta \, d\alpha = \alpha \frac{d\beta}{\beta} - d\alpha$$

A solution of the equation $\Lambda = 0$ is consequently given by

$$\frac{\alpha}{\beta} = \text{const.} \quad \text{or} \quad ze^{\frac{z}{y}} = \text{const.}$$

2. Reduce the expression

$$(ax + by + cz) \, dx + (lx + my + nz) \, dy + (px + qy + rz) \, dz$$

to the canonical form, $a, b, c, l, m, n, p, q, r$ being constants.

3. Solve the equations

$$yz \, dx + zx \, dy + dz = 0$$

$$(y - z) \, dx + (z - x) \, dy + (x - y) \, dz = 0$$

$$x(y - 1)(z - 1) \, dx + y(z - 1)(x - 1) \, dy + z(x - 1)(y - 1) \, dz$$

4. Prove that two differential forms

$$\Lambda \equiv P \, dx + Q \, dy + R \, dz \qquad \Lambda' \equiv P' \, dx + Q' \, dy + R' \, dz$$

can be reduced to the forms

$$\Lambda = U \, du + V \, dv \qquad \Lambda' = U' \, du + V' \, dv$$

respectively, where $u = \text{const.}$, $v = \text{const.}$ are two independent solutions of the equations

$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}$$

5. Show that, in the last example, we may write

$$\begin{aligned}\lambda U &= (PP' + QQ' + RR') \left(P \frac{\partial v}{\partial x} + Q \frac{\partial v}{\partial y} + R \frac{\partial v}{\partial z} \right) \\ &\quad - (P^2 + Q^2 + R^2) \left(P' \frac{\partial v}{\partial x} + Q' \frac{\partial v}{\partial y} + R' \frac{\partial v}{\partial z} \right) \\ \lambda V &= (P^2 + Q^2 + R^2) \left(P' \frac{\partial u}{\partial x} + Q' \frac{\partial u}{\partial y} + R' \frac{\partial u}{\partial z} \right) \\ &\quad - (PP' + QQ' + RR') \left(P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} \right) \\ \lambda U' &= (PP' + QQ' + RR') \left(P' \frac{\partial v}{\partial x} + Q' \frac{\partial v}{\partial y} + R' \frac{\partial v}{\partial z} \right) \\ &\quad - (P'^2 + Q'^2 + R'^2) \left(P \frac{\partial v}{\partial x} + Q \frac{\partial v}{\partial y} + R \frac{\partial v}{\partial z} \right) \\ \lambda V' &= (P'^2 + Q'^2 + R'^2) \left(P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} \right) \\ &\quad - (PP' + QQ' + RR') \left(P' \frac{\partial u}{\partial x} + Q' \frac{\partial u}{\partial y} + R' \frac{\partial u}{\partial z} \right)\end{aligned}$$

where λ is some function of x, y, z .

6. Prove that, with the notation of the last two examples, the functions

$$\theta = \frac{U'}{U} \quad \varphi = \frac{V'}{V}$$

are solutions of the partial differential equation

$$\begin{aligned}(QR' - Q'R) \frac{\partial \theta}{\partial x} + (RP' - R'P) \frac{\partial \theta}{\partial y} + (PQ' - P'Q) \frac{\partial \theta}{\partial z} \\ = P'\xi' + Q'\eta' + R'\zeta' - \theta(P\xi' + Q\eta' + R\zeta' + P'\xi + Q'\eta + R'\zeta) \\ + \theta^2(P\xi + Q\eta + R\zeta)\end{aligned}$$

where $\xi' = \frac{\partial R'}{\partial y} - \frac{\partial Q'}{\partial z}$, etc.

§ 61. *Homogeneous equations.* If in the equation $\Lambda = 0$, P, Q, R are homogeneous and of the same degree in x, y, z , the equation $\Lambda = 0$ can be written in the form

$$dx + F\left(\frac{y}{x}, \frac{z}{x}\right) dy + G\left(\frac{y}{x}, \frac{z}{x}\right) dz = 0$$

Putting $y = sx, z = tx$, the equation becomes

$$dx[1 + sF(s, t) + tG(s, t)] + xF(s, t) ds + xG(s, t) dt = 0$$

or
$$\frac{dx}{x} + \frac{F ds + G dt}{1 + sF + tG} = 0$$

Now, if the invariant of the first equation vanishes, the invariant of this equation will also vanish, and this means that the preceding equation can be integrated as it stands.

Example. Let us solve the equation

$$(x^2 - y^2 - z^2) dx + 2xy dy + 2xz dz = 0$$

by this method. Putting $y = xs$, $z = xt$, we get

$$\frac{dx}{x} + 2 \frac{s ds + t dt}{1 + s^2 + t^2} = 0$$

$$\therefore x(1 + s^2 + t^2) = c$$

or

$$x^2 + y^2 + z^2 = cx$$

EXAMPLES.

1. Solve the equations

$$2yz dx + 2zx dy - xy dz = 0$$

$$(2x + y + z)(y + z) dx + (2y + z + x)(z + x) dy + (2z + x + y)(x + y) dz = 0$$

$$(y - z) dx + (z - x) dy + (x - y) dz = 0$$

$$3(2y - 3z)(8yz - x^2) dx + 12x(x^2 + 4z^2) dy - 2x(9x^2 + 16y^2) dz = 0$$

2. Solve the equation

$$dx + (e^{z-x} - 1) dy + (e^{y-x} - 1) dz = 0$$

CHAPTER VIII

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

§ 62. *The homogeneous linear partial differential equation of the second order.* The general theory of partial differential equations of the second order is beyond the scope of the present treatise; there are, however, special types of equations of frequent occurrence which can be treated briefly by elementary methods, and these will now be considered.

If A, B, C, D, E, F are functions of x and y only, the equation

$$A \frac{\partial^2 z}{\partial x^2} + 2B \frac{\partial^2 z}{\partial x \partial y} + C \frac{\partial^2 z}{\partial y^2} + D \frac{\partial z}{\partial x} + E \frac{\partial z}{\partial y} + Fz = 0 \quad (1)$$

is called a homogeneous linear partial differential equation of the second order.

In attempting to solve this equation the first step is to reduce it to a canonical form; we accordingly make the substitution

$$x = f(\xi, \eta) \quad y = g(\xi, \eta) \quad \dots \quad (2)$$

and take ξ, η as new independent variables. The transformed equation is

$$A' \frac{\partial^2 z}{\partial \xi^2} + 2B' \frac{\partial^2 z}{\partial \xi \partial \eta} + C' \frac{\partial^2 z}{\partial \eta^2} + D' \frac{\partial z}{\partial \xi} + E' \frac{\partial z}{\partial \eta} + Fz = 0 \quad (3)$$

$$\text{where } \left. \begin{aligned} A' &= A \left(\frac{\partial \xi}{\partial x} \right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2 \\ B' &= A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \\ C' &= A \left(\frac{\partial \eta}{\partial x} \right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2 \\ D' &= A \frac{\partial^2 \xi}{\partial x^2} + 2B \frac{\partial^2 \xi}{\partial x \partial y} + C \frac{\partial^2 \xi}{\partial y^2} + D \frac{\partial \xi}{\partial x} + E \frac{\partial \xi}{\partial y} \\ E' &= A \frac{\partial^2 \eta}{\partial x^2} + 2B \frac{\partial^2 \eta}{\partial x \partial y} + C \frac{\partial^2 \eta}{\partial y^2} + D \frac{\partial \eta}{\partial x} + E \frac{\partial \eta}{\partial y} \end{aligned} \right\} \quad (4)$$

If $B^2 - AC$ and A are different from zero, the quadratic equation

$$A\lambda^2 - 2B\lambda + C = 0 \quad . \quad . \quad . \quad (5)$$

has two different roots λ_1, λ_2 , and we may write

$$A\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2B\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + C\left(\frac{\partial\varphi}{\partial y}\right)^2 = A\left(\frac{\partial\varphi}{\partial x} + \lambda_1\frac{\partial\varphi}{\partial y}\right)\left(\frac{\partial\varphi}{\partial x} + \lambda_2\frac{\partial\varphi}{\partial y}\right).$$

Let $\xi(x, y)$ and $\eta(x, y)$ satisfy the equations

$$\frac{\partial\xi}{\partial x} + \lambda_1\frac{\partial\xi}{\partial y} = 0$$

$$\frac{\partial\eta}{\partial x} + \lambda_2\frac{\partial\eta}{\partial y} = 0$$

respectively ; then $\frac{\partial(\xi, \eta)}{\partial(x, y)}$ is not zero, and so we may regard ξ and η as independent variables. Let the transformation

$$\xi = \xi(x, y) \quad \eta = \eta(x, y)$$

be equivalent to (2) ; then for the new equation (3), $A' = C' = 0$, while $B' \neq 0$. Dividing by $2B'$, equation (3) takes the *canonical* form

$$\frac{\partial^2 z}{\partial\xi\partial\eta} + a\frac{\partial z}{\partial\xi} + b\frac{\partial z}{\partial\eta} + cz = 0 \quad . \quad . \quad . \quad (6)$$

When the roots of equation (5) are both real, equation (1) is said to be *hyperbolic* ; when the roots are both imaginary, the equation (1) is said to be *elliptic* ; when the equation (5) has equal roots, equation (1) is said to be *parabolic*. These terms were used in this connection by P. du Bois-Reymond. The transformation was given by Laplace in 1773.

The last case will now be discussed ; it evidently occurs when $B^2 = AC$.

Writing $Au^2 + 2Buv + Cv^2 \equiv A(u + \lambda v)^2$

we have $A' = A\left(\frac{\partial\xi}{\partial x} + \lambda\frac{\partial\xi}{\partial y}\right)^2$

$$B' = A\left(\frac{\partial\xi}{\partial x} + \lambda\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial x} + \lambda\frac{\partial\eta}{\partial y}\right)$$

$$C' = A\left(\frac{\partial\eta}{\partial x} + \lambda\frac{\partial\eta}{\partial y}\right)^2$$

where the functions $\xi(x, y)$ and $\eta(x, y)$ are at present arbitrary. Now let η be chosen so that

$$\frac{\partial\eta}{\partial x} + \lambda\frac{\partial\eta}{\partial y} = 0$$

and ξ so that

$$\frac{\partial\xi}{\partial x} + \lambda\frac{\partial\xi}{\partial y} \neq 0$$

then $B' = C' = 0$, while $A' \neq 0$. Dividing by A' , the transformed equation (3) takes the canonical form

$$\frac{\partial^2 z}{\partial \xi^2} + a \frac{\partial z}{\partial \xi} + b \frac{\partial z}{\partial \eta} + cz = 0 \quad (7)$$

If $A = 0$ and $B \neq 0$, we interchange x and y and proceed as before; if both A and B are zero, the differential equation (1) is already in the canonical form.

In discussing the properties of partial differential equations of the second order, it is convenient to write

$$r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad t = \frac{\partial^2 z}{\partial y^2} \quad p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}$$

If we are given a series of elements (x, y, z, p, q) depending on a parameter θ and satisfying the relation $dz = p dx + q dy$, the second derivatives r, s, t are connected by the relations

$$dp = r dx + s dy \quad dq = s dx + t dy$$

These equations, together with the partial differential equation (1), generally enable us to determine the values of r, s, t uniquely; for if we eliminate r and t respectively, we get

$$\left. \begin{aligned} s(A dy^2 - 2B dx dy + C dx^2) &= A dp dy + C dq dx \\ &+ (Dp + Eq + Fz) dx dy \end{aligned} \right\} \quad (8)$$

This equation determines s , and the other two equations determine r and t when s is known.

If, however, the equation

$$A dy^2 - 2B dx dy + C dx^2 = 0 \quad (9)$$

is satisfied, there is no finite value of s unless we also have

$$A dp dy + C dq dx + (Dp + Eq + Fz) dx dy = 0$$

When this last equation holds, the equation (8) does not determine s at all, and ∞' sets of values of r, s, t consistent with the given series of elements (x, y, z, p, q) .

The equation (9) is called *the differential equation of the characteristics* of equation (1). If $\theta(x, y) = \text{constant}$ is consistent with

(9), we have $\frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = 0$, and so

$$A \left(\frac{\partial \theta}{\partial x} \right)^2 + 2B \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} + C \left(\frac{\partial \theta}{\partial y} \right)^2 = 0 \quad (10)$$

This is another form for the equation of the characteristics; it will be seen that the functions ξ, η , which were used in obtaining the canonical form (6), are solutions of this *partial differential equation of the characteristics*.

If we regard (x, y) as the coordinates of a point in a plane, the equation (9) can be regarded as the differential equation of a family of curves; the equation (9) evidently specifies two possible directions for the tangent to one of these curves.

Let the curves defined by the differential equation (9) be called *characteristic curves*, or simply *characteristics*; then, by comparing equation (9) with (5), we see that a hyperbolic equation has real characteristics, an elliptic equation imaginary characteristics, and a parabolic equation coincident characteristics.

It should be noticed that if the partial differential equation (1) possesses a solution of the form

$$z = \gamma f(\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where θ, γ are particular functions of x, y and f an arbitrary function, θ must satisfy the partial differential equation (10). In general, however, the equation (1) does not possess a solution of this form. To see this, let us consider the equation in the canonical form (6), and write

$$z = \gamma f(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The differential equation will then be satisfied if

$$\begin{aligned} f(\xi) \left[\frac{\partial^2 \gamma}{\partial \xi \partial \eta} + a \frac{\partial \gamma}{\partial \xi} + b \frac{\partial \gamma}{\partial \eta} + c \gamma \right] + f'(\xi) \left[\frac{\partial \gamma}{\partial \eta} + a \gamma \right] &= 0 \\ \text{i.e. if} \quad \left. \begin{aligned} \frac{\partial^2 \gamma}{\partial \xi \partial \eta} + a \frac{\partial \gamma}{\partial \xi} + b \frac{\partial \gamma}{\partial \eta} + c \gamma &= 0 \\ \frac{\partial \gamma}{\partial \eta} + a \gamma &= 0 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (13) \end{aligned}$$

Differentiating the last equation with regard to ξ , we find that

$$b \frac{\partial \gamma}{\partial \eta} + \left(c - \frac{\partial a}{\partial \xi} \right) \gamma = 0$$

hence the two equation (13) can only coexist if

$$ab = c - \frac{\partial a}{\partial \xi} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This, then, is the condition that the differential equation (6) should possess a solution of type (12).

When this condition is satisfied, the partial differential equation may be written in the form

$$\frac{\partial}{\partial \xi} \left(\frac{\partial z}{\partial \eta} + az \right) + b \left(\frac{\partial z}{\partial \eta} + az \right) = 0$$

Integrating this equation by the method of § 7, we get

$$\frac{\partial z}{\partial \eta} + az = \psi(\eta) e^{-\int b a d\xi}$$

This equation may now be solved by a second application of the method of § 7, and we get

$$z = f(\xi)e^{-\int a d\eta} + e^{-\int a d\eta} \left[\int \psi(\eta) d\eta \cdot \exp \left\{ \int a d\eta - b d\xi \right\} \right]$$

We evidently have $\gamma = e^{-\int a d\eta}$

§ 63. *Laplace's method.* The preceding method may be generalised by writing equation (6) in the forms

$$\frac{\partial z_1}{\partial \xi} + bz_1 - hz = 0 \quad z_1 = \frac{\partial z}{\partial \eta} + az \quad . \quad (15)$$

$$\frac{\partial z_{-1}}{\partial \eta} + az_{-1} - kz = 0 \quad z_{-1} = \frac{\partial z}{\partial \xi} + bz \quad . \quad (16)$$

where $h = \frac{\partial a}{\partial \xi} + ab - c$

$$k = \frac{\partial b}{\partial \eta} + ab - c$$

If either $h = 0$ or $k = 0$, the equation may be integrated by the preceding method; if not, it is sometimes useful to form the partial differential equations satisfied by z_1 and z_{-1} . These are

$$\frac{\partial^2 z_1}{\partial \xi \partial \eta} + a_1 \frac{\partial z_1}{\partial \xi} + b_1 \frac{\partial z_1}{\partial \eta} + c_1 z_1 = 0 \quad . \quad (17)$$

$$\frac{\partial^2 z_{-1}}{\partial \xi \partial \eta} + a_{-1} \frac{\partial z_{-1}}{\partial \xi} + b_{-1} \frac{\partial z_{-1}}{\partial \eta} + c_{-1} z_{-1} = 0 \quad . \quad (18)$$

respectively, where

$$a_1 = a - \frac{\partial}{\partial \eta}(\log h) \quad b_1 = b$$

$$c_1 = c - \frac{\partial a}{\partial \xi} + \frac{\partial b}{\partial \eta} - b \frac{\partial}{\partial \eta}(\log h)$$

$$a_{-1} = a \quad b_{-1} = b - \frac{\partial}{\partial \xi}(\log k)$$

$$c_{-1} = c - \frac{\partial b}{\partial \eta} + \frac{\partial a}{\partial \xi} - a \frac{\partial}{\partial \xi}(\log k)$$

Writing

$$h_1 = \frac{\partial a_1}{\partial \xi} + a_1 b_1 - c_1 \quad k_1 = \frac{\partial b_1}{\partial \eta} + a_1 b_1 - c_1$$

$$h_{-1} = \frac{\partial a_{-1}}{\partial \xi} + a_{-1} b_{-1} - c_{-1} \quad k_{-1} = \frac{\partial b_{-1}}{\partial \eta} + a_{-1} b_{-1} - c_{-1}$$

we have $h_1 = 2h - k - \frac{\partial^2}{\partial \xi \partial \eta}(\log h) \quad k_1 = h$

$$h_{-1} = k \quad k_{-1} = 2k - h - \frac{\partial^2}{\partial \xi \partial \eta}(\log k)$$

If either h_1 or k_{-1} is zero, one of the new partial differential equations can be solved by the preceding method. If neither is zero, the transformation may be repeated, and this process continued a number of times until a soluble equation is obtained, or until it is realised that the method is unsuitable for the equation in hand.

It should be noticed that if equation (17) is transformed by putting

$$\frac{\partial z_1}{\partial \xi} + b_1 z_1 = z'$$

we have

$$z' = hz$$

The equation satisfied by z' is thus derivable from (6) by a simple transformation of the dependent variable, and will be regarded as equivalent to (6), because the quantities h', k' for the new equation are equal to h, k respectively. In fact, if we make the substitution

$$z = \lambda(\xi, \eta) z'$$

the equation satisfied by z' is

$$\frac{\partial^2 z'}{\partial \xi \partial \eta} + a' \frac{\partial z'}{\partial \xi} + b' \frac{\partial z'}{\partial \eta} + c' z' = 0$$

where

$$a' = a + \frac{\partial}{\partial \eta} (\log \lambda)$$

$$b' = b + \frac{\partial}{\partial \xi} (\log \lambda)$$

$$c' = c + a \frac{\partial}{\partial \xi} (\log \lambda) + b \frac{\partial}{\partial \eta} (\log \lambda) + \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial \xi \partial \eta}$$

Thus

$$h' = \frac{\partial a'}{\partial \xi} + a' b' - c' = h$$

$$k' = \frac{\partial b'}{\partial \eta} + a' b' - c' = k$$

The quantities h', k' are on this account called the invariants of the differential equation, and the operation by which h_{-1}, k_{-1} are derived from h, k is regarded as the inverse of the operation by which h_1, k_1 are derived from h, k .

For further information on this subject the reader is referred to Darboux, *Théorie des Surfaces*, Vol. II.

EXAMPLES.

1. Solve the equations

$$(x - y)s + p - q = 0$$

$$x(r - t) + 2p = 0$$

2. Show that the equation

$$r^{r+1} \frac{\partial^2 u}{\partial p^2} = a^2 \frac{\partial^2 u}{\partial q^2}$$

can be reduced to the form

$$(\xi + \eta) \frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{3 - \gamma}{2(\gamma - 1)} \left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \right)$$

by the substitution

$$u = pv \quad \xi = \frac{2a}{\gamma - 1} p^{\frac{1-\gamma}{2}} + q \quad \eta = \frac{2a}{\gamma - 1} p^{\frac{1-\gamma}{2}} - q$$

Hence show that when $\gamma = 1.4$,

$$v = (\xi + \eta)^2 \varphi''(\xi) - 6(\xi + \eta) \varphi'(\xi) + 12\varphi(\xi) + (\xi + \eta)^2 \psi''(\eta) - 6(\xi + \eta) \psi'(\eta) + 12\psi(\eta)$$

where φ and ψ are arbitrary functions.

3. Prove that the equation of propagation of plane waves of sound, viz.

$$\left(\frac{\partial y}{\partial x} \right)^{\gamma+1} \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

can be reduced to the linear equation

$$p^{\gamma+1} \frac{\partial^2 u}{\partial p^2} = a^2 \frac{\partial^2 u}{\partial q^2}$$

by the substitution

$$\frac{\partial y}{\partial x} = p \quad \frac{\partial y}{\partial t} = q \quad u = px + qt - y \quad (\text{J. R. Wilton.})$$

4. Reduce the parabolic equation

$$\frac{\partial^2 z}{\partial x^2} + a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + cz + f = 0$$

to the canonical form $\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial y} = pw + q$

and find the condition that p may be zero.

(A. R. Forsyth and H. Block.)

§ 64. *D'Alembert's equation.* Let us consider the small transverse vibrations of a fine inextensible string of uniform line-density ρ when the vibrations take place in one plane and the action of gravity is neglected. Let the axis of x be taken along the equilibrium position of the string, and let the centre of mass of an element of the string of length δx have coordinates x, y at time t .

Since the vibrations are transverse, we may regard $\frac{d^2 x}{dt^2}$ as negligible; consequently if T denote the tension of the string at one end of the element, the equations of motion are

$$\begin{aligned} \delta(T \cos \psi) &= 0 \\ \delta(T \sin \psi) &= \rho \delta x \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

where ψ is the angle which the tangent at a point of the curve formed by the string makes with the axis of x , and the symbol δ is used to denote an increment of the quantity following it.

The first equation indicates that $T \cos \psi$ is a constant quantity T_0 . If in the second equation we neglect quantities of the second order, it is legitimate to regard (x, y) as the coordinates of a point on the curve formed by the string, and to write $\tan \psi = \frac{\partial y}{\partial x}$, where y is regarded as a function of x and t .

Assuming that $\frac{\partial^2 y}{\partial x^2}$ is continuous along the element in question, we may write

$$\delta(T \sin \psi) = \delta \left(T_0 \frac{\partial y}{\partial x} \right) = T_0 \frac{\partial^2 y}{\partial x^2} \delta x$$

when quantities of the second order are neglected. We thus obtain the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots \quad (1)$$

where $c^2 = \frac{T_0}{\rho}$ is a positive quantity independent of x and t .

Equation (1) is usually called *d'Alembert's equation*; it was the first partial differential equation to be considered, and is of historical interest on account of the discussion which took place as to the nature of the general solution.

To reduce equation (1) to the canonical form, we must solve the differential equation of the characteristics, namely

$$dx^2 - c^2 dt^2 = 0 \quad \dots \quad (2)$$

Two independent solutions of this equation are

$$x - ct = \text{const.} \quad \text{and} \quad x + ct = \text{const.}$$

respectively. Let us write $\alpha = x - ct$, $\beta = x + ct$; then equation (1) is reduced to the canonical form

$$\frac{\partial^2 y}{\partial \alpha \partial \beta} = 0 \quad \dots \quad (3)$$

and its general integral is seen to be

$$y = F(\alpha) + G(\beta) \quad \dots \quad (4)$$

where F and G are arbitrary functions. Substituting the values of α and β , we get $y = F(x - ct) + G(x + ct) \quad \dots \quad (5)$

The supplementary conditions suggested by the physical problem may take various forms. If the position and transverse velocity of each portion of the string is known at time $t = 0$, the supplementary conditions are

$$y = f(x) \quad \frac{\partial y}{\partial t} = g(x) \quad \text{when } t = 0 \quad \dots \quad (6)$$

The value of y given by (5) will satisfy the requirements if

$$\begin{aligned} f(x) &= F(x) + G(x) \\ g(x) &= cF'(x) - cG'(x) \end{aligned}$$

Integrating the last equation, we obtain

$$\frac{1}{c} \int_0^x g(\xi) d\xi = F(x) - G(x) + A$$

where A is some constant whose value is not needed.

We now find that

$$2F(x + ct) = f(x + ct) + \frac{1}{c} \int_0^{x+ct} g(\xi) d\xi - A$$

$$2G(x - ct) = f(x - ct) - \frac{1}{c} \int_0^{x-ct} g(\xi) d\xi + A$$

$$\text{Hence } y = \frac{1}{2} \left[f(x + ct) + f(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(\xi) d\xi \right]. \quad (7)$$

This formula gives the required solution when the second derivative of $f(\xi)$ and the first derivative of $g(\xi)$ exist, and are finite for $\xi = x - ct + \epsilon$ and $\xi = x + ct - \epsilon$, where ϵ is an arbitrarily small positive quantity.

Now let us consider the case when

$$\begin{aligned} f(x) &= 0 & g(x) &= 0 \text{ for } x > a \text{ and } x < a_1 \\ & & g(x) &\neq 0 \text{ for } a_1 < x < a \end{aligned}$$

so that only a limited portion of the string has an initial velocity. The solution is now expressed by a different formula in different cases.

If $t > 0$ and $x - ct > a$, the value of $g(\xi)$ appearing in the integral in formula (7) is zero for the whole range of integration.

$$\therefore y = 0 \text{ when } x > ct$$

If $t > 0$ and $a_1 \leq x - ct < a$ while $x + ct < a$, we have

$$y = \frac{1}{2c} \int_{x-ct}^a g(\xi) d\xi$$

but if $x + ct < a$, we have

$$y = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi$$

If $t > 0$ and $a_1 > x - ct$ while $x + ct < a$, we have

$$y = \frac{1}{2c} \int_{a_1}^{x+ct} g(\xi) d\xi$$

If $t > 0$, $a_1 > x - ct$ and $x + ct > a$, we have

$$y = \frac{1}{2c} \int_{a_1}^a g(\xi) d\xi$$

Finally, if $x + ct < a_1$, we have $y = 0$.

It follows from this result that two waves travel out from the region of initial disturbance in opposite directions with velocity c . Before a wave reaches a point x there is no disturbance. Each wave

is of thickness $a - a_1$. Some points are influenced by both waves and others by only one.

When the supplementary conditions state that $y = 0$ for $x = 0$, and for $x = l$ whatever value t may have, and that

$$y = f(x) \quad \frac{\partial y}{\partial t} = g(x) \text{ for } t = 0, \quad 0 \leq x \leq l$$

the solution is not given immediately by formula (7), because the values of $f(\xi)$ and $g(\xi)$ are unknown when ξ lies outside the interval $0 \leq \xi \leq l$. We may, however, proceed as follows :

Since $y = 0$ when $x = 0$, formula (7) gives

$$f(ct) + f(-ct) + \frac{1}{c} \int_{-ct}^{ct} g(\xi) d\xi = 0$$

This equation must hold for all values of t ; writing $-t$ in place of t , we get

$$f(ct) + f(-ct) + \frac{1}{c} \int_{ct}^{-ct} g(\xi) d\xi = 0$$

$$\therefore \left. \begin{aligned} f(ct) + f(-ct) &= 0 \\ g(ct) + g(-ct) &= 0 \end{aligned} \right\} \dots \dots \dots (8)$$

These equations enable us to determine the values of $f(\xi)$ and $g(\xi)$ for negative values of ξ when their values for positive values of ξ are known. In particular, we can deduce their values in the interval $-l \leq \xi \leq 0$.

Again, when $x = l$, $y = 0$, and formula (7) gives

$$f(l+ct) + f(l-ct) + \frac{1}{c} \int_{l-ct}^{l+ct} g(\xi) d\xi = 0$$

This formula must hold for all values of t ; let us write $l-ct$ in place of $l+ct$, then we must also write $l+ct$ in place of $l-ct$, for

$$2l - (l-ct) = l+ct \quad 2l - (l+ct) = l-ct$$

We now get

$$f(l-ct) + f(l+ct) + \frac{1}{c} \int_{l+ct}^{l-ct} g(\xi) d\xi = 0$$

hence it follows that

$$\left. \begin{aligned} f(l-ct) + f(l+ct) &= 0 \\ g(l-ct) + g(l+ct) &= 0 \end{aligned} \right\} \dots \dots \dots (9)$$

These equations connect the values of f and g at two points equidistant from the point $x = l$. With the aid of the two sets of equations (8) and (9), we can define $f(x)$ and $g(x)$ for all real values of x in terms of their values in the interval $(0 \leq x \leq l)$, and the solution of our problem is then given by formula (7). The result can be expressed concisely as follows :

$f(x)$ and $g(x)$ are odd periodic functions of period $2l$.

Another solution of this problem will be obtained in Chapter IX by the method of infinite series.

EXAMPLES.

1. A function $V = f(x, t)$ is such that the sum of its values at one pair of opposite corners of a parallelogram is equal to the sum of its values at the other pair of opposite corners, the parallelogram being any parallelogram whose sides are parallel to the lines $x - ct = 0$, $x + ct = 0$ respectively. Prove that if the second derivatives of the function f exist and are finite, V satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 V}{\partial t^2}$$

2. Let

$$u = f(x + ct) + f(x - ct)$$

$$v = f(x + ct) - f(x - ct)$$

and let P be a parallelogram whose sides are parallel to the lines $x - ct = 0$, $x + ct = 0$. Prove that if a curve $u = \text{const.}$ pass through one pair of opposite corners of P , a curve $v = \text{const.}$ will pass through the other pair of opposite corners.

3. If the supplementary conditions state that

$$f(x) = A \sin \frac{\pi x}{l} \quad g(x) = B \sin \frac{\pi x}{l} \quad y = 0 \text{ for } x = 0, x = l$$

prove that the appropriate solution of $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is

$$y = A \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + \frac{Bl}{\pi} \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l}$$

§ 65. *The equation of wave-propagation in three dimensions. Maxwell's equations.* The partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} \quad \dots \quad (1)$$

is a natural extension of D'Alembert's equation. It is fundamental in the theories of the propagation of waves of light and sound.* In particular, the quantities $\alpha, \beta, \gamma, X, Y, Z$ in Maxwell's equations †

$$\left. \begin{aligned} \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} &= \frac{1}{c} \frac{\partial X}{\partial t} & \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} &= -\frac{1}{c} \frac{\partial \alpha}{\partial t} \\ \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} &= \frac{1}{c} \frac{\partial Y}{\partial t} & \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} &= -\frac{1}{c} \frac{\partial \beta}{\partial t} \\ \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} &= \frac{1}{c} \frac{\partial Z}{\partial t} & \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} &= -\frac{1}{c} \frac{\partial \gamma}{\partial t} \\ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} &= 0 & \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} &= 0 \end{aligned} \right\} \quad \dots \quad (2)$$

* The constant c represents the velocity of light or the velocity of sound, as the case may be. The equation was, I believe, first given by Euler.

† Cf. J. J. Thomson, *Elements of Electricity and Magnetism*, p. 484. In the present form of the equations, X, Y, Z are the components of the electric intensity,

for the propagation of electric waves are all solutions of (1). To prove this, we remark that

$$\begin{aligned} & \frac{\partial}{\partial x} \left[\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right] - \frac{\partial}{\partial y} \left[\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} - \frac{1}{c} \frac{\partial Z}{\partial t} \right] \\ & + \frac{\partial}{\partial z} \left[\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} - \frac{1}{c} \frac{\partial Y}{\partial t} \right] + \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} - \frac{1}{c} \frac{\partial \alpha}{\partial t} \right] \\ & \equiv \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \alpha}{\partial t^2} \end{aligned}$$

Hence α satisfies equation (1), and in a similar way it can be shown that β, γ, X, Y, Z satisfy equation (1).

Now let us endeavour to satisfy equations (2) by expressions of type

$$\begin{aligned} \alpha &= \alpha_0 f(u, v) & \beta &= \beta_0 f(u, v) & \gamma &= \gamma_0 f(u, v) \\ X &= X_0 f(u, v) & Y &= Y_0 f(u, v) & Z &= Z_0 f(u, v) \end{aligned}$$

where $\alpha_0, \beta_0, \gamma_0, X_0, Y_0, Z_0, u, v$ are certain functions of x, y, z, t and f is an arbitrary function of u and v . Substituting in the first of equations (2), we get

$$\begin{aligned} f \left[\frac{\partial \gamma_0}{\partial y} - \frac{\partial \beta_0}{\partial z} - \frac{1}{c} \frac{\partial X_0}{\partial t} \right] + \frac{\partial f}{\partial u} \left[\gamma_0 \frac{\partial u}{\partial y} - \beta_0 \frac{\partial u}{\partial z} - \frac{1}{c} X_0 \frac{\partial u}{\partial t} \right] \\ + \frac{\partial f}{\partial v} \left[\gamma_0 \frac{\partial v}{\partial y} - \beta_0 \frac{\partial v}{\partial z} - \frac{1}{c} X_0 \frac{\partial v}{\partial t} \right] = 0 \end{aligned}$$

Since f is arbitrary, we must have

$$\begin{aligned} \gamma_0 \frac{\partial u}{\partial y} - \beta_0 \frac{\partial u}{\partial z} - \frac{1}{c} X_0 \frac{\partial u}{\partial t} &= 0 \\ \gamma_0 \frac{\partial v}{\partial y} - \beta_0 \frac{\partial v}{\partial z} - \frac{1}{c} X_0 \frac{\partial v}{\partial t} &= 0 \\ \frac{\partial \gamma_0}{\partial y} - \frac{\partial \beta_0}{\partial z} - \frac{1}{c} \frac{\partial X_0}{\partial t} &= 0 \end{aligned}$$

The first two equations give

$$\gamma_0 = \lambda \frac{\partial(u, v)}{\partial(z, t)} \quad \beta_0 = \lambda \frac{\partial(u, v)}{\partial(y, t)} \quad X_0 = -c\lambda \frac{\partial(u, v)}{\partial(y, z)}$$

while the third equation gives

$$\frac{\partial(\lambda, u, v)}{\partial(y, z, t)} = 0$$

and α, β, γ are *proportional* to the components of the magnetic force, the multiplier being $\sqrt{\mu/\kappa}$, where κ is the specific inductive capacity and μ the permeability of the medium. Also $c^2 = \frac{1}{\kappa\mu}$.

Treating the other equations in a similar way, we find that we must have

$$\begin{aligned} Z_0 &= \mu \frac{\partial(u, v)}{\partial(z, t)} & Y_0 &= \mu \frac{\partial(u, v)}{\partial(y, t)} & \alpha_0 &= c\mu \frac{\partial(u, v)}{\partial(y, z)} \\ \alpha_0 &= \nu \frac{\partial(u, v)}{\partial(x, t)} & \gamma_0 &= \nu \frac{\partial(u, v)}{\partial(z, t)} & Y_0 &= -c\nu \frac{\partial(u, v)}{\partial(z, x)} \end{aligned}$$

and a number of similar equations. Comparing these with the previous equations, we easily find that $\nu = \lambda$, and that

$$\left. \begin{aligned} \alpha_0 &= \lambda \frac{\partial(u, v)}{\partial(x, t)} = \mu c \frac{\partial(u, v)}{\partial(y, z)} & X_0 &= -c\lambda \frac{\partial(u, v)}{\partial(y, z)} = \mu \frac{\partial(u, v)}{\partial(x, t)} \\ \beta_0 &= \lambda \frac{\partial(u, v)}{\partial(y, t)} = \mu c \frac{\partial(u, v)}{\partial(z, x)} & Y_0 &= -c\lambda \frac{\partial(u, v)}{\partial(z, x)} = \mu \frac{\partial(u, v)}{\partial(y, t)} \\ \gamma_0 &= \lambda \frac{\partial(u, v)}{\partial(z, t)} = \mu c \frac{\partial(u, v)}{\partial(x, y)} & Z_0 &= -c\lambda \frac{\partial(u, v)}{\partial(x, y)} = \mu \frac{\partial(u, v)}{\partial(z, t)} \end{aligned} \right\} \quad (3)$$

Comparing these equations, we find that $\lambda^2 + \mu^2 = 0$.

$$\therefore \mu = i\lambda \quad (i = \sqrt{-1}).$$

There are also four equations of type

$$\frac{\partial(\lambda, u, v)}{\partial(y, z, t)} = \frac{\partial(\lambda, u, v)}{\partial(z, x, t)} = \frac{\partial(\lambda, u, v)}{\partial(x, y, t)} = \frac{\partial(\lambda, u, v)}{\partial(x, y, z)} = 0$$

which indicate that λ is a function of u and v ; we may without loss of generality take it to be unity.

We now have the result that formulae (3) give a set of solutions of equations (2), provided the functions u and v satisfy the relations

$$\left. \begin{aligned} \frac{\partial(u, v)}{\partial(x, t)} &= ic \frac{\partial(u, v)}{\partial(y, z)} \\ \frac{\partial(u, v)}{\partial(y, t)} &= ic \frac{\partial(u, v)}{\partial(z, x)} \\ \frac{\partial(u, v)}{\partial(z, t)} &= ic \frac{\partial(u, v)}{\partial(x, y)} \end{aligned} \right\} \quad \dots \dots \dots (4)$$

To solve these equations we take u, v, x, y as new independent variables and write the equations in the form

$$\begin{aligned} \frac{\partial(u, v, y, z)}{\partial(x, t, y, z)} &= ic \frac{\partial(u, v, x, t)}{\partial(y, z, x, t)} \\ \frac{\partial(u, v, z, x)}{\partial(y, t, z, x)} &= ic \frac{\partial(u, v, y, t)}{\partial(z, x, y, t)} \\ \frac{\partial(u, v, x, y)}{\partial(z, t, x, y)} &= ic \frac{\partial(u, v, z, t)}{\partial(x, y, z, t)} \end{aligned}$$

Now multiply both sides of each equation by $\frac{\partial(x, y, z, t)}{\partial(u, v, x, y)}$; then the theorem relating to the multiplication of Jacobians gives

$$\frac{\partial(u, v, y, z)}{\partial(u, v, x, y)} = ic \frac{\partial(u, v, x, t)}{\partial(u, v, x, y)}$$

$$\frac{\partial(u, v, z, x)}{\partial(u, v, x, y)} = ic \frac{\partial(u, v, y, t)}{\partial(u, v, x, y)}$$

$$\frac{\partial(u, v, x, y)}{\partial(u, v, x, y)} = ic \frac{\partial(u, v, z, t)}{\partial(u, v, x, y)}$$

Evaluating each functional determinant, we find that the transformed equations are

$$\left. \begin{aligned} -\frac{\partial z}{\partial x} &= ic \frac{\partial t}{\partial y} \\ -\frac{\partial z}{\partial y} &= -ic \frac{\partial t}{\partial x} \\ 1 &= ic \frac{\partial(z, t)}{\partial(x, y)} \end{aligned} \right\} \dots \dots \dots (5)$$

The first two equations give

$$\frac{\partial}{\partial x}(z - ct) = i \frac{\partial}{\partial y}(z - ct)$$

$$\frac{\partial}{\partial x}(z + ct) = -i \frac{\partial}{\partial y}(z + ct)$$

Solving these equations by Lagrange's method, we get

$$z - ct = F(x - iy, u, v) \quad z + ct = G(x + iy, u, v)$$

$$\text{Now} \quad 2 \frac{\partial z}{\partial x} = F_1 + G_1 \quad 2c \frac{\partial t}{\partial x} = G_1 - F_1$$

$$2 \frac{\partial z}{\partial y} = i(G_1 - F_1) \quad 2c \frac{\partial t}{\partial y} = i(F_1 + G_1)$$

where F_1 denotes the derivative of F with regard to its first argument; hence

$$4ic \frac{\partial(z, t)}{\partial(x, y)} = (G_1 - F_1)^2 - (F_1 + G_1)^2 = -4F_1G_1.$$

$$\therefore F_1G_1 = -1.$$

This equation can only be satisfied if both F_1 and G_1 are independent of x and y ; consequently we may write

$$\left. \begin{aligned} z - ct &= \varphi(u, v) + \theta(u, v)(x - iy) \\ z + ct &= \psi(u, v) - \frac{x + iy}{\theta(u, v)} \end{aligned} \right\} \dots \dots (6)$$

where θ, φ, ψ are arbitrary functions of u and v . If u and v are defined by these equations, the relations (4) are satisfied.

The preceding equations can be written in the form

$$z - \zeta - c(t - \tau) = \theta[x - \xi - i(y - \eta)]$$

$$z - \zeta + c(t - \tau) = -\frac{1}{\theta}[x - \xi + i(y - \eta)]$$

where ξ, η, ζ, τ are functions of u and v such that

$$\zeta - c\tau = \theta(\xi - i\eta) + \varphi$$

$$\zeta + c\tau = -\frac{1}{\theta}(\xi + i\eta) + \psi$$

If we eliminate θ , we obtain the equation

$$(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = c^2(t - \tau)^2$$

Two independent functions of u and v will evidently serve our purpose as well as u and v , and it is usually convenient to write

$$u = \tau \quad v = \theta$$

If, then, θ and τ are defined by the equations

$$\left. \begin{aligned} (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 &= c^2(t - \tau)^2 \\ z - \zeta - c(t - \tau) &= \theta[x - \xi - i(y - \eta)] \end{aligned} \right\} \quad (7)$$

where ξ, η, ζ are arbitrary functions of θ and τ , we can say that the expressions

$$\begin{aligned} \alpha &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(x, t)} & X &= -cf(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(y, z)} \\ \beta &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(y, t)} & Y &= -cf(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(z, x)} \\ \gamma &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(z, t)} & Z &= -cf(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(x, y)} \end{aligned}$$

will furnish a set of solutions of equations (2), and will all be solutions of equation (1). To obtain a real set of solutions, we may retain only the real parts of the above expressions.

In the particular case when ξ, η, ζ are functions of τ only, we have

$$\begin{aligned} y - \eta &= M \frac{\partial \tau}{\partial y} & z - \zeta &= M \frac{\partial \tau}{\partial z} \\ \frac{\partial \theta}{\partial y} [x - \xi - i(y - \eta)] - i\theta + N \frac{\partial \tau}{\partial y} &= 0 \\ \frac{\partial \theta}{\partial z} [x - \xi - i(y - \eta)] - 1 + N \frac{\partial \tau}{\partial z} &= 0 \end{aligned}$$

$$\text{where } M = \frac{\partial \xi}{\partial \tau}(x - \xi) + \frac{\partial \eta}{\partial \tau}(y - \eta) + \frac{\partial \zeta}{\partial \tau}(z - \zeta) - c^2(t - \tau) \quad (8)$$

$$\text{and } N = \frac{\partial \zeta}{\partial \tau} - c - \theta \left(\frac{\partial \xi}{\partial \tau} - i\eta \right)$$

Hence

$$M[x - \xi - i(y - \eta)] \frac{\partial(\theta, \tau)}{\partial(y, z)} = \left| \begin{array}{cc} i\theta & 1 \\ y - \eta & z - \zeta \end{array} \right| = i\theta(z - \zeta) - (y - \eta)$$

but
$$z - \zeta = \frac{1}{2} \left(\theta - \frac{1}{\theta} \right) (x - \xi) - \frac{i}{2} \left(\theta + \frac{1}{\theta} \right) (y - \eta)$$

$$\therefore i\theta(z - \zeta) - (y - \eta) = \frac{i}{2}(\theta^2 - 1)[x - \xi - i(y - \eta)]$$

and so
$$\frac{\partial(\theta, \tau)}{\partial(y, z)} = \frac{i}{2M}(\theta^2 - 1)$$

Writing $g(\theta, \tau) = (\theta^2 - 1)f(\theta, \tau)$, we have the result that the function

$$V = \frac{1}{M} g(\theta, \tau) \quad \dots \dots \dots (9)$$

satisfies the partial differential equation (1).

When $\xi = \eta = \zeta = 0$, equations (7) and (8) give

$$\tau = t - \frac{r}{c} \quad \theta = \frac{z - r}{x - iy} \quad M = -cr$$

where

$$r^2 = x^2 + y^2 + z^2$$

Hence the function

$$V = \frac{1}{r} g \left(t - \frac{r}{c}, \frac{z - r}{x - iy} \right) \quad \dots \dots \dots (10)$$

satisfies the partial differential equation (1), g being an arbitrary function.

This result may be generalised by replacing x, y, z by $x - \xi, y - \eta, z - \zeta$ respectively, where ξ, η, ζ are arbitrary constants; this is, in fact, what is obtained when we take $\xi(\tau), \eta(\tau), \zeta(\tau)$ to be constant instead of zero. We may also change the sign of r in the expression (10) and obtain a new solution.

The particular solutions

$$V = \frac{1}{r} g \left(t - \frac{r}{c} \right) \quad \text{and} \quad V = \frac{1}{M} g(\tau)$$

are of considerable importance in mathematical physics.

EXAMPLES.

1. If u is defined by the equation

$$xl(u) + ym(u) + zn(u) - ct p(u) = q(u)$$

where

$$l^2 + m^2 + n^2 = p^2$$

prove that $V = f(u)$ satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

and that

$$\frac{g(u)}{x \frac{dl}{du} + y \frac{dm}{du} + z \frac{dn}{du} - ct \frac{dp}{du} - \frac{dq}{du}}$$

is also a solution, l, m, n, q, f, g being arbitrary functions of u .

(A. R. Forsyth.)

2. If θ is a constant and f an arbitrary function, the function

$$V = f[x \cos \theta + y \sin \theta + iz, x \sin \theta - y \cos \theta - ct]$$

satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

3. Prove that two functions u, v which satisfy equations (4) of § 65 also satisfy the equations

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 &= \frac{1}{c^2} \left(\frac{\partial u}{\partial t}\right)^2 \\ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 &= \frac{1}{c^2} \left(\frac{\partial v}{\partial t}\right)^2 \\ \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} &= \frac{1}{c^2} \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \end{aligned}$$

4. Prove that if u and v satisfy equations (4) of § 65, the ratios of the Jacobians

$$\frac{\partial(u, v)}{\partial(y, z)} \quad \frac{\partial(u, v)}{\partial(z, x)} \quad \frac{\partial(u, v)}{\partial(x, y)}$$

are functions of u and v .

5. Prove that equations (2) may be satisfied by writing

$$\begin{aligned} \alpha &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} & \beta &= \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} & \gamma &= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \\ X &= -\frac{\partial V}{\partial x} - \frac{1}{c} \frac{\partial F}{\partial t} & Y &= -\frac{\partial V}{\partial y} - \frac{1}{c} \frac{\partial G}{\partial t} & Z &= -\frac{\partial V}{\partial z} - \frac{1}{c} \frac{\partial H}{\partial t} \end{aligned}$$

where F, G, H, V are solutions of the equation (1) and are connected by the relation

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{1}{c} \frac{\partial V}{\partial t} = 0$$

6. If $u = \alpha + iX, \quad v = \beta + iY, \quad w = \gamma + iZ, \quad i = \sqrt{-1}$

Maxwell's equations can be written in the form

$$\begin{aligned} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= -\frac{i}{c} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= -\frac{i}{c} \frac{\partial w}{\partial t} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= -\frac{i}{c} \frac{\partial v}{\partial t} & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}$$

(H. Weber and L. Silberstein.)

7. Prove that the last set of equations can be satisfied by writing

$$\begin{aligned} u &= i \left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} \right) = \frac{1}{c} \frac{\partial F}{\partial t} + \frac{\partial \Lambda}{\partial x} \\ v &= i \left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} \right) = \frac{1}{c} \frac{\partial G}{\partial t} + \frac{\partial \Lambda}{\partial y} \\ w &= i \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) = \frac{1}{c} \frac{\partial H}{\partial t} + \frac{\partial \Lambda}{\partial z} \end{aligned}$$

where

$$\begin{aligned} F &= \frac{1}{c} \frac{\partial L}{\partial t} + i \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \right) + \frac{\partial K}{\partial x} \\ G &= \frac{1}{c} \frac{\partial M}{\partial t} + i \left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \right) + \frac{\partial K}{\partial y} \\ H &= \frac{1}{c} \frac{\partial N}{\partial t} + i \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) + \frac{\partial K}{\partial z} \\ \Lambda &= - \frac{\partial L}{\partial x} - \frac{\partial M}{\partial y} - \frac{\partial N}{\partial z} - \frac{1}{c} \frac{\partial K}{\partial t} \end{aligned}$$

and L, M, N, K are solutions of equation (1). (Hertz and Righi.)

8. Prove that if x_0, y_0, z_0, t_0 are constants, the function

$$V = \frac{1}{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - c^2(t - t_0)^2}$$

satisfies equation (1).

9. If in Maxwell's equations $\alpha = \beta = \gamma = 0$, so that there is no magnetic force, X, Y, Z are independent of t , and

$$X dx + Y dy + Z dz$$

is a complete differential $-d\Phi$. Hence show that the potential Φ in an electrostatic field satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

10. Prove that, with the notation of Ex. 5, the conditions are satisfied by

$$F = \frac{\xi'(\tau)}{M} \quad G = \frac{\eta'(\tau)}{M} \quad H = \frac{\zeta'(\tau)}{M} \quad V = \frac{c}{M}$$

where M and τ are the quantities defined above. When

$$\xi'^2 + \eta'^2 + \zeta'^2 < c^2$$

this solution corresponds to a moving point charge of electricity.

(A. Liénard.)

§ 65a. *The fundamental equations of the theory of electrons.* These equations are the same as equations (2) of § 65, except that the first set of four equations is replaced by the following set of equations: *

$$\left. \begin{aligned} \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} &= \frac{1}{c} \frac{\partial X}{\partial t} + \frac{\rho u}{c} & \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} &= \frac{1}{c} \frac{\partial Z}{\partial t} + \frac{\rho w}{c} \\ \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} &= \frac{1}{c} \frac{\partial Y}{\partial t} + \frac{\rho v}{c} & \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} &= \rho \end{aligned} \right\} \quad (A)$$

The quantity ρ is called the volume density of electricity, and (u, v, w) are regarded as the component velocities of the element of

* Cf. H. A. Lorentz, *The Theory of Electrons*, Ch. I. If ρ, u, v, w are given, it is easy to see that each of the six quantities $\alpha, \beta, \gamma, X, Y, Z$ satisfies a partial differential equation of the second order.

electricity which is at the point x, y, z at time t . The second set of four equations given in § 65 is retained. A particular set of solutions of these equations of the form

$$\begin{aligned}\alpha &= \alpha_0 f(\theta, \tau) & \beta &= \beta_0 f(\theta, \tau) & \gamma &= \gamma_0 f(\theta, \tau) \\ X &= X_0 f(\theta, \tau) & Y &= Y_0 f(\theta, \tau) & Z &= Z_0 f(\theta, \tau)\end{aligned}$$

may be found as before. The appropriate expressions are :

$$\left. \begin{aligned}\alpha &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(y, z)} & X &= f(\theta, \tau) \frac{1}{c} \frac{\partial(\theta, \tau)}{\partial(x, t)} \\ \beta &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(z, x)} & Y &= f(\theta, \tau) \frac{1}{c} \frac{\partial(\theta, \tau)}{\partial(y, t)} \\ \gamma &= f(\theta, \tau) \frac{\partial(\theta, \tau)}{\partial(x, y)} & Z &= f(\theta, \tau) \frac{1}{c} \frac{\partial(\theta, \tau)}{\partial(z, t)}\end{aligned} \right\} \quad \text{I}$$

where θ and τ are at present arbitrary functions of x, y, z, t and $f(\theta, \tau)$ is an arbitrary function of θ and τ . It should be noticed that these expressions make $\alpha X + \beta Y + \gamma Z \equiv 0$, so the solution is certainly of a particular type.

When the above expressions are substituted in equations (A), a new set of equations are obtained, which are easily solved if ρ, u, v, w are regarded as the unknowns, but which are quite difficult to solve if θ and τ are regarded as the unknowns.

Let us suppose that the paths of the elements of electricity are given by the equations

$$\left. \begin{aligned}\varphi(x, y, z, t) &= \text{const.} & \psi(x, y, z, t) &= \text{const.} \\ \chi(x, y, z, t) &= \text{const.}\end{aligned} \right\} \quad \text{(B)}$$

These equations may be regarded as integrals of the system of differential equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt$$

consequently φ, ψ and χ satisfy the partial differential equations

$$u \frac{\partial \varphi}{\partial x} + v \frac{\partial \varphi}{\partial y} + w \frac{\partial \varphi}{\partial z} + \frac{\partial \varphi}{\partial t} = 0$$

$$u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} + w \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial t} = 0$$

$$u \frac{\partial \chi}{\partial x} + v \frac{\partial \chi}{\partial y} + w \frac{\partial \chi}{\partial z} + \frac{\partial \chi}{\partial t} = 0$$

Solving these equations for u, v, w , we find that

$$ku = \frac{\partial(\varphi, \psi, \chi)}{\partial(y, z, t)} \quad kv = \frac{\partial(\varphi, \psi, \chi)}{\partial(z, x, t)} \quad kw = \frac{\partial(\varphi, \psi, \chi)}{\partial(x, y, t)}$$

where

$$k = - \frac{\partial(\varphi, \psi, \chi)}{\partial(x, y, z)}$$

Now equations (A) give the relation

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0$$

and can only be solved when this equation is satisfied. Substituting the above values of u, v, w , the relation takes the form

$$\frac{\partial(\omega, \varphi, \psi, \chi)}{\partial(x, y, z, t)} = 0$$

where $\omega = \frac{\rho}{k}$. Hence $\frac{\rho}{k}$ is a function of φ, ψ, χ , and so we may write

$$\rho = - \frac{\partial(\varphi, \psi, \chi)}{\partial(x, y, z)} G(\varphi, \psi, \chi)$$

If $\varphi_1, \psi_1, \chi_1$ are functions of φ, ψ, χ such that

$$\frac{\partial(\varphi_1, \psi_1, \chi_1)}{\partial(\varphi, \psi, \chi)} = G(\varphi, \psi, \chi)$$

the preceding equation can be written in the form

$$\rho = - \frac{\partial(\varphi_1, \psi_1, \chi_1)}{\partial(x, y, z)}$$

Let us take these functions $\varphi_1, \psi_1, \chi_1$ for our functions φ, ψ, χ , which specify the paths of the particles of electricity; then we may put $G = 1$.

Now let us endeavour to find a solution of equations (A) which gives another solution when $\alpha, \beta, \gamma, X, Y, Z, \rho$ are multiplied by the same arbitrary function of two parameters p, q , and u, v, w are left unaltered.

Substituting the expressions

$$\begin{array}{llll} \alpha_1 = \alpha F(p, q) & \beta_1 = \beta F(p, q) & \gamma_1 = \gamma F(p, q) & \\ X_1 = X F(p, q) & Y_1 = Y F(p, q) & Z_1 = Z F(p, q) & \rho_1 = \rho F(p, q) \end{array}$$

in the equations

$$\begin{array}{ll} \frac{\partial \gamma_1}{\partial y} - \frac{\partial \beta_1}{\partial z} = \frac{1}{c} \frac{\partial X_1}{\partial t} + \frac{\rho_1 u}{c} & \frac{\partial \beta_1}{\partial x} - \frac{\partial \alpha_1}{\partial y} = \frac{1}{c} \frac{\partial Z_1}{\partial t} + \frac{\rho_1 w}{c} \\ \frac{\partial \alpha_1}{\partial z} - \frac{\partial \gamma_1}{\partial x} = \frac{1}{c} \frac{\partial Y_1}{\partial t} + \frac{\rho_1 v}{c} & \frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y} + \frac{\partial Z_1}{\partial z} = \rho_1 \end{array}$$

and proceeding as in § 65, we find that we must have

$$\begin{array}{ll} \alpha = \lambda \frac{\partial(p, q)}{\partial(x, t)} & X = - c\lambda \frac{\partial(p, q)}{\partial(y, z)} \\ \beta = \lambda \frac{\partial(p, q)}{\partial(y, t)} & Y = - c\lambda \frac{\partial(p, q)}{\partial(z, x)} \\ \gamma = \lambda \frac{\partial(p, q)}{\partial(z, t)} & Z = - c\lambda \frac{\partial(p, q)}{\partial(x, y)} \end{array}$$

$$\begin{aligned}\rho u &= c \frac{\partial(\lambda, p, q)}{\partial(y, z, t)} & \rho v &= c \frac{\partial(\lambda, p, q)}{\partial(z, x, t)} \\ \rho w &= c \frac{\partial(\lambda, p, q)}{\partial(x, y, t)} & \rho &= -c \frac{\partial(\lambda, p, q)}{\partial(x, y, z)}\end{aligned}$$

Comparing these expressions for ρ, u, v, w with those obtained previously, we see that the two sets may be made equivalent by writing

$$p = \varphi \quad q = \psi \quad c\lambda = \chi$$

Hence, when the paths of the elements of electricity are given by equations of type (B), a set of solutions of equations (A) is obtained by writing

$$\left. \begin{aligned}\alpha &= \frac{\chi}{c} \frac{\partial(\varphi, \psi)}{\partial(x, t)} \\ \beta &= \frac{\chi}{c} \frac{\partial(\varphi, \psi)}{\partial(y, t)} \\ \gamma &= \frac{\chi}{c} \frac{\partial(\varphi, \psi)}{\partial(z, t)}\end{aligned} \right\} \begin{aligned}X &= -\chi \frac{\partial(\varphi, \psi)}{\partial(y, z)} \\ Y &= -\chi \frac{\partial(\varphi, \psi)}{\partial(z, x)} \\ Z &= -\chi \frac{\partial(\varphi, \psi)}{\partial(x, y)}\end{aligned} \quad \dots \quad \text{II}$$

A more general solution is obtained by adding to these expressions for $\alpha, \beta, \gamma, X, Y, Z$, the corresponding expressions for a set of solutions of equations A when $\rho = 0$. Since

$$\rho = -\frac{\partial(\varphi, \psi, \chi)}{\partial(x, y, z)}$$

we may make ρ zero by taking χ to be a function of φ and ψ . Hence we may add to the above expressions quantities of type

$$\alpha = \sum_s f_s(\varphi_s, \psi_s) \frac{\partial(\varphi_s, \psi_s)}{\partial(x, t)} \quad X = -\sum_s c f_s(\varphi_s, \psi_s) \frac{\partial(\varphi_s, \psi_s)}{\partial(y, z)}$$

where φ_s and ψ_s are arbitrary functions of x, y, z, t and $f_s(\varphi_s, \psi_s)$ is an arbitrary function of φ_s and ψ_s . The summation indicates that a number of different solutions of the type under consideration are added together.*

The particular solution given by equations II is of special interest for several reasons. In the first place, the equations

$$\varphi(x, y, z, t) = \text{const.} \quad \psi(x, y, z, t) = \text{const.}$$

may be regarded as the equations of a *line of electric force*, i.e. a curve which moves in such a way that the tangent at any point

* A much simpler set of solutions of equations A, when $\rho = 0$, is obtained by writing

$$\begin{aligned}X &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & \alpha &= \frac{1}{c} \frac{\partial P}{\partial t} + \frac{\partial S}{\partial x} \\ Y &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} & \beta &= \frac{1}{c} \frac{\partial Q}{\partial t} + \frac{\partial S}{\partial y} \\ Z &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} & \gamma &= \frac{1}{c} \frac{\partial R}{\partial t} + \frac{\partial S}{\partial z}\end{aligned}$$

where P, Q, R, S are arbitrary functions of x, y, z and t .

of it at any instant is in the direction of the electric intensity at that point. To see this, we keep t constant and vary x, y, z ; then the direction of the tangent of the curves is given by

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz = 0$$

and so
$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

This proves the theorem.

Secondly, if we combine II with (B), it is easy to see that the relations

$$c\alpha = vZ - wY \quad c\beta = wX - uZ \quad c\gamma = uY - vX$$

are satisfied. These are the relations adopted in Sir Joseph Thomson's representation of an electromagnetic field by means of a single set of moving tubes of force.* It should be noticed also that $\alpha X + \beta Y + \gamma Z = 0$.

This means that the electric intensity is at right angles to the magnetic force, a relation which also holds in the case of the solution (I) of the other set of four equations. Since we want to obtain a set of quantities $\alpha, \beta, \gamma, X, Y, Z$ satisfying the two sets of four equations, we shall endeavour to find the conditions that the two sets of expressions (I) and (II) may be equivalent.

Equations (I) give

$$\alpha \frac{\partial \tau}{\partial x} + \beta \frac{\partial \tau}{\partial y} + \gamma \frac{\partial \tau}{\partial z} = 0 \quad \frac{\partial \tau}{\partial z} X - \frac{\partial \tau}{\partial x} Z - \frac{\partial \tau}{\partial t} \frac{\beta}{c} = 0$$

$$\frac{\partial \tau}{\partial y} Z - \frac{\partial \tau}{\partial z} Y - \frac{\partial \tau}{\partial t} \frac{\alpha}{c} = 0 \quad \frac{\partial \tau}{\partial x} Y - \frac{\partial \tau}{\partial y} X - \frac{\partial \tau}{\partial t} \frac{\gamma}{c} = 0$$

Substituting the expressions for $\alpha, \beta, \gamma, X, Y, Z$ given in II. and writing

$$\Gamma = \frac{\partial \tau}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial \tau}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial \tau}{\partial z} \frac{\partial \varphi}{\partial z} - \frac{1}{c^2} \frac{\partial \tau}{\partial t} \frac{\partial \varphi}{\partial t}$$

$$\Lambda = \frac{\partial \tau}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \tau}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \tau}{\partial z} \frac{\partial \psi}{\partial z} - \frac{1}{c^2} \frac{\partial \tau}{\partial t} \frac{\partial \psi}{\partial t}$$

we find that either $\chi = 0$ (which is a trivial case) or

$$\Gamma \frac{\partial \psi}{\partial t} - \Lambda \frac{\partial \varphi}{\partial t} = 0 \quad \Gamma \frac{\partial \psi}{\partial y} - \Lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\Gamma \frac{\partial \psi}{\partial x} - \Lambda \frac{\partial \varphi}{\partial x} = 0 \quad \Gamma \frac{\partial \psi}{\partial z} - \Lambda \frac{\partial \varphi}{\partial z} = 0$$

If we suppose that $\alpha, \beta, \gamma, X, Y, Z$ are not zero, these equations give

$$\Gamma = 0 \quad \Lambda = 0$$

* *Recent Researches* (1893), Ch. I.

In a similar way we find that

$$\frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial\theta}{\partial z}\frac{\partial\varphi}{\partial z} - \frac{1}{c^2}\frac{\partial\theta}{\partial t}\frac{\partial\varphi}{\partial t} = 0$$

$$\frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\theta}{\partial z}\frac{\partial\psi}{\partial z} - \frac{1}{c^2}\frac{\partial\theta}{\partial t}\frac{\partial\psi}{\partial t} = 0$$

If φ and ψ are given, θ and τ must be chosen so as to satisfy these four equations. As a matter of fact, the equations are not soluble unless φ and ψ satisfy certain conditions which have not yet been expressed in a simple form. If we regard (x, y, z, ict) as rectangular coordinates in a space of four dimensions, the four equations imply that the surfaces $\theta = \text{const.}$, $\tau = \text{const.}$ are orthogonal to the surfaces $\varphi = \text{const.}$, $\psi = \text{const.}$ In particular, the conditions are satisfied if the varieties $\theta = \text{const.}$, $\tau = \text{const.}$, $\varphi = \text{const.}$, $\psi = \text{const.}$ form a mutually orthogonal system.

EXAMPLES.

1. Prove that if τ and θ are defined by the equations

$$(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = c^2(t - \tau)^2$$

$$\theta = \frac{f(x - \xi) + g(y - \eta) + h(z - \zeta) + k}{\xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta) - c^2(t - \tau)}$$

where $\xi, \eta, \zeta, f, g, h, k$ are arbitrary functions of τ and primes denote differentiations with respect to τ ; then φ and ψ will satisfy the four equations at the end of § 65a if they are taken to be the real and imaginary parts of the expression

$$F\left[\frac{x - \xi + i(y - \eta)}{z - \zeta + c(t - \tau)}, \tau\right]$$

where $F[\sigma, \tau] = \text{const.}$ is a solution of the Riccetan equation

$$2k\frac{d\sigma}{d\tau} = [\xi' + i\eta' - \sigma(\zeta' + c)][\sigma(f - ig) + h] \\ - [\sigma(\xi' - i\eta') + \zeta' - c][f + ig - \sigma h]$$

2. Let the expressions I be formed with the values of θ and τ given in the last example, and let ξ, η, ζ be regarded as the coordinates at time τ of a moving point E. Prove that if particles are shot out from the different positions of E, and travel along straight lines with the velocity of light, they will form at each instant a line of electric force if the direction cosines (l, m, n) of the direction of projection at time τ satisfy the differential equations

$$k\frac{dl}{d\tau} = f(c - l\xi' - m\eta' - n\zeta') + (\xi' - cl)(fl + gm + hn)$$

$$k\frac{dm}{d\tau} = g(c - l\xi' - m\eta' - n\zeta') + (\eta' - cm)(fl + gm + hn)$$

$$k\frac{dn}{d\tau} = h(c - l\xi' - m\eta' - n\zeta') + (\zeta' - cn)(fl + gm + hn)$$

Prove that the solution of these equations may be made to depend on that of the Riccattian equation considered in the last example.

3. If in the last example $f(0, \tau)$ is a constant, the velocity of the electricity is the same as if particles of electricity were shot out in all directions from the moving point E and travelled along straight lines with the velocity of light.

(The case when $k = c^2 - \xi'^2 - \eta'^2 - \zeta'^2$ and is positive is of special interest, because $\alpha, \beta, \gamma, X, Y, Z, \rho, u, v, w$ are all finite except at the singular point E, and there is a constant electric charge associated with the moving point E, so that equal quantities of positive and negative electricity are shot out from it at each instant.)

4. Prove that if, in the last example,

$$f = \xi'' \quad g = \eta'' \quad h = \zeta'' \quad k = c^2 - \xi'^2 - \eta'^2 - \zeta'^2$$

the quantity ρ and, consequently, also $\rho u, \rho v, \rho w$ are zero.

The solution then becomes identical with that of Ex. 10 of § 65, and so represents a point charge of electricity. (R. Hargreaves.)

5. Prove that the fundamental equations may be solved by writing

$$\begin{aligned} \alpha &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} & X &= -\frac{1}{c} \frac{\partial F}{\partial t} - \frac{\partial V}{\partial x} \\ \beta &= \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} & Y &= -\frac{1}{c} \frac{\partial G}{\partial t} - \frac{\partial V}{\partial y} \\ \gamma &= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} & Z &= -\frac{1}{c} \frac{\partial H}{\partial t} - \frac{\partial V}{\partial z} \end{aligned}$$

where F, G, H, V are solutions of the equations

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} + \frac{\rho u}{c} &= 0 \\ \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} + \frac{\rho v}{c} &= 0 \\ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} + \frac{\rho w}{c} &= 0 \\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} + \rho &= 0 \\ \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{1}{c} \frac{\partial V}{\partial t} &= 0 \end{aligned}$$

6. Prove that if the fundamental equations possess a solution of type

$$\begin{aligned} \alpha &= \alpha_0 f(\tau) & \beta &= \beta_0 f(\tau) & \gamma &= \gamma_0 f(\tau) \\ X &= X_0 f(\tau) & Y &= Y_0 f(\tau) & Z &= Z_0 f(\tau) \\ \rho &= \rho_0 f(\tau) + \rho_1 f'(\tau) \end{aligned}$$

representing a series of waves or independent pulses, and $f(\tau)$ is arbitrary on account of the independence of the separate pulses originating from the source, the phase τ satisfies the partial differential equation

$$\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial y}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{1}{c^2} \left(\frac{\partial \tau}{\partial t}\right)^2$$

Hence show that waves or pulses of this type are propagated with the velocity of light. Prove also that $\alpha X + \beta Y + \gamma Z = 0$.

7. If in Ex. 2 particles shot out from one position of E are associated with particles shot out from a consecutive position of E by means of the transformation

$$\begin{aligned}x_0 &= x + \frac{d\tau}{k}[\xi'L - fM] & y_0 &= y + \frac{d\tau}{k}[\eta'L - gM] \\z_0 &= z + \frac{d\tau}{k}[\zeta'L - hM] & t_0 &= t + \frac{d\tau}{k}L\end{aligned}$$

where L and M are the numerator and denominator respectively of the expression for θ given in Ex. 1, then corresponding particles are associated with the same line of electric force. Hence show that a particle which moves according to the law

$$\frac{dx}{dt} = \xi' - \frac{f}{\theta} \quad \frac{dy}{dt} = \eta' - \frac{g}{\theta} \quad \frac{dz}{dt} = \zeta' - \frac{h}{\theta}$$

will always lie on the same line of electric force.

8. Prove that a solution of equations A and the supplementary equations $ca = vZ - wY$, $c\beta = wX - uZ$, $c\gamma = uY - vX$, is necessarily of the form II.

§ 66. *Laplace's equation.* The partial differential equation $\nabla^2 V = 0$ or

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad . \quad . \quad (1)$$

is of fundamental importance in electrostatics, hydrodynamics, and the theory of attractions. It evidently arises when we attempt to find a solution of the equation of wave propagation which does not depend upon t . Now we have already seen that when f is an arbitrary function, $\frac{1}{r}f\left(t - \frac{r}{c}\right)$ is a solution of the equation of wave propagation; consequently, if we make f constant, we find that $V = \frac{1}{r}$ is a solution of equation (1). This is true if

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 \quad . \quad . \quad . \quad (2)$$

where ξ, η, ζ are arbitrary constants which can be regarded as the coordinates of a point P; r is then the distance of the point (x, y, z) from P.

The differential equation (1) is usually called *Laplace's equation*, and a certain type of solution of this equation is called a *potential function* * or *harmonic function*. When V is independent of z , it satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad . \quad . \quad . \quad (3)$$

* The potential function occurs in electrostatics and the theory of attractions as the function representing the sum of a number of terms of type $\frac{e}{r}$, where e depends only on ξ, η, ζ , and the summation extends over a number of different points of type P.

which may be reduced to the canonical form

$$\frac{\partial^2 V}{\partial \alpha \partial \beta} = 0$$

by writing $x + iy = \alpha$, $x - iy = \beta$. The complete solution of (3) is consequently

$$V = f(x + iy) + g(x - iy) \quad (4)$$

where f and g are arbitrary functions.

It is evident that if $u + iv = f(x + iy)$, $u - iv = f(x - iy)$, then u and v are solutions of Laplace's equation; they also satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and are called *conjugate functions*.

Laplace's equation (1) evidently possesses particular solutions of the form

$$V = u(x) v(y) w(z) \quad (5)$$

for if we substitute this expression for V , we obtain

$$u(x) v(y) w(z) \left[\frac{1}{u} \frac{d^2 u}{dx^2} + \frac{1}{v} \frac{d^2 v}{dy^2} + \frac{1}{w} \frac{d^2 w}{dz^2} \right] = 0$$

Hence, if we write

$$\frac{1}{u} \frac{d^2 u}{dx^2} = l^2 \quad \frac{1}{v} \frac{d^2 v}{dy^2} = m^2 \quad \frac{1}{w} \frac{d^2 w}{dz^2} = n^2$$

where $l^2 + m^2 + n^2 = 0$, equation (1) will be satisfied. The preceding equations may be satisfied by taking

$$u = e^{lx} \quad v = e^{my} \quad w = e^{nz}$$

l, m, n being constants. Hence

$$V = e^{lx+my+nz} \quad (6)$$

is a particular solution of the required type. The usefulness of particular solutions of equation (1) is increased by the fact that since the equation is linear, the sum of any number of solutions is also a solution. We have, in fact,

$$\nabla^2(U + V + W + \dots) = \nabla^2 U + \nabla^2 V + \nabla^2 W + \dots$$

which is zero if $U, V, W \dots$ are solutions of Laplace's equation.

To generalise the solution (5), we may consider the problem of finding solutions of Laplace's equation of type

$$V = u(\alpha) v(\beta) w(\gamma) \quad (7)$$

where α, β, γ are certain functions of x, y, z which are supposed to be given. The natural method of solving this problem is to take α, β, γ as new independent variables and to find the new form of Laplace's equation. The transformation may be effected as follows.

Let $dx, dy, dz; \delta x, \delta y, \delta z$ be two independent sets of increments of the variables $x, y, z; d\alpha, d\beta, d\gamma; \delta\alpha, \delta\beta, \delta\gamma$, the corresponding sets of increments of the variables α, β, γ ; then, since

$$dx = \frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta + \frac{\partial x}{\partial \gamma} d\gamma \quad \delta x = \frac{\partial x}{\partial \alpha} \delta\alpha + \frac{\partial x}{\partial \beta} \delta\beta + \frac{\partial x}{\partial \gamma} \delta\gamma, \text{ etc.}$$

we find that *

$$\left. \begin{aligned} dx \delta x + dy \delta y + dz \delta z &= a d\alpha \delta\alpha + b d\beta \delta\beta + c d\gamma \delta\gamma \\ &+ f(d\beta \delta\gamma + d\gamma \delta\beta) + g(d\gamma \delta\alpha + \delta\gamma d\alpha) + h(d\alpha \delta\beta + \delta\alpha d\beta) \end{aligned} \right\} \quad (8)$$

where a, b, c, f, g, h are functions of α, β, γ . If now we have a relation of type

$$\left. \begin{aligned} l d(y, z) + m d(z, x) + n d(x, y) \\ = p d(\beta, \gamma) + q d(\gamma, \alpha) + r d(\alpha, \beta) \end{aligned} \right\} \quad (9)$$

we may deduce by multiplication, using the symbolical method of § 57, that

$$\begin{aligned} (l \delta x + m \delta y + n \delta z) d(x, y, z) \\ = [(ap + hq + gr) \delta\alpha + (hp + bq + fr) \delta\beta \\ + (gp + fq + cr) \delta\gamma] d(\alpha, \beta, \gamma) \end{aligned}$$

Now write $d(x, y, z) = J d(\alpha, \beta, \gamma)$; then the above equation is equivalent to the identical relation

$$\frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y + \frac{\partial V}{\partial z} \delta z = \frac{\partial V}{\partial \alpha} \delta\alpha + \frac{\partial V}{\partial \beta} \delta\beta + \frac{\partial V}{\partial \gamma} \delta\gamma$$

$$\text{if} \quad l = \frac{\partial V}{\partial x} \quad m = \frac{\partial V}{\partial y} \quad n = \frac{\partial V}{\partial z}$$

$$ap + hq + gr = J \frac{\partial V}{\partial \alpha}$$

$$hp + bq + fr = J \frac{\partial V}{\partial \beta}$$

$$gp + fq + cr = J \frac{\partial V}{\partial \gamma}$$

Solving for p, q, r , we get

$$\Delta p = J \left[A \frac{\partial V}{\partial \alpha} + H \frac{\partial V}{\partial \beta} + G \frac{\partial V}{\partial \gamma} \right]$$

$$\Delta q = J \left[H \frac{\partial V}{\partial \alpha} + G \frac{\partial V}{\partial \beta} + F \frac{\partial V}{\partial \gamma} \right]$$

$$\Delta r = J \left[G \frac{\partial V}{\partial \alpha} + F \frac{\partial V}{\partial \beta} + C \frac{\partial V}{\partial \gamma} \right]$$

* Notice also that

$$dx^2 + dy^2 + dz^2 = a da^2 + b d\beta^2 + c d\gamma^2 + 2f d\beta d\gamma + 2g d\gamma da + 2h da d\beta$$

It is generally convenient to determine a, b, c, f, g, h by writing down this equation.

where A, B, C, F, G, H are the minors of a, b, c, f, g, h in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

To calculate the value of J , we notice that if we equate coefficients of $\delta\alpha, \delta\beta, \delta\gamma$ in (8), we get

$$dx \frac{\partial x}{\partial \alpha} + dy \frac{\partial y}{\partial \alpha} + dz \frac{\partial z}{\partial \alpha} = a d\alpha + h d\beta + g d\gamma$$

$$dx \frac{\partial x}{\partial \beta} + dy \frac{\partial y}{\partial \beta} + dz \frac{\partial z}{\partial \beta} = h d\alpha + b d\beta + f d\gamma$$

$$dx \frac{\partial x}{\partial \gamma} + dy \frac{\partial y}{\partial \gamma} + dz \frac{\partial z}{\partial \gamma} = g d\alpha + f d\beta + c d\gamma$$

Multiplying these equations symbolically, we get

$$d(x, y, z) \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)} = \Delta d(\alpha, \beta, \gamma)$$

$$\text{but } d(x, y, z) = J d(\alpha, \beta, \gamma), \quad \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)} = J$$

therefore

$$J^2 = \Delta$$

Substituting the values of l, m, n, p, q, r in (9), we have the identity

$$\begin{aligned} \frac{\partial V}{\partial x} d(y, z) + \frac{\partial V}{\partial y} d(z, x) + \frac{\partial V}{\partial z} d(x, y) &= \frac{A \frac{\partial V}{\partial \alpha} + H \frac{\partial V}{\partial \beta} + G \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} d(\beta, \gamma) \\ &+ \frac{H \frac{\partial V}{\partial \alpha} + B \frac{\partial V}{\partial \beta} + F \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} d(\gamma, \alpha) + \frac{G \frac{\partial V}{\partial \alpha} + F \frac{\partial V}{\partial \beta} + C \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} d(\alpha, \beta) \end{aligned}$$

Differentiating this equation by the symbolical method, we obtain

$$\begin{aligned} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) d(x, y, z) &= d(\alpha, \beta, \gamma) \left[\frac{\partial}{\partial \alpha} \left(\frac{A \frac{\partial V}{\partial \alpha} + H \frac{\partial V}{\partial \beta} + G \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \right. \\ &+ \frac{\partial}{\partial \beta} \left(\frac{H \frac{\partial V}{\partial \alpha} + B \frac{\partial V}{\partial \beta} + F \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \\ &\left. + \frac{\partial}{\partial \gamma} \left(\frac{G \frac{\partial V}{\partial \alpha} + F \frac{\partial V}{\partial \beta} + C \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \right] \end{aligned}$$

Hence *

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\sqrt{\Delta}} \left[\frac{\partial}{\partial \alpha} \left(\frac{A \frac{\partial V}{\partial \alpha} + H \frac{\partial V}{\partial \beta} + G \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \right. \\ \left. + \frac{\partial}{\partial \beta} \left(\frac{H \frac{\partial V}{\partial \alpha} + B \frac{\partial V}{\partial \beta} + F \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \right. \\ \left. + \frac{\partial}{\partial \gamma} \left(\frac{G \frac{\partial V}{\partial \alpha} + F \frac{\partial V}{\partial \beta} + C \frac{\partial V}{\partial \gamma}}{\sqrt{\Delta}} \right) \right] \quad (10) \end{aligned}$$

In particular, if

$$dx \delta x + dy \delta y + dz \delta z = \lambda^2 d\alpha \delta \alpha + \mu^2 d\beta \delta \beta + \nu^2 d\gamma \delta \gamma \quad (11)$$

we have

$$\Delta = \lambda^2 \mu^2 \nu^2 \quad A = \mu^2 \nu^2 \quad B = \nu^2 \lambda^2 \quad C = \lambda^2 \mu^2 \quad F = G = H = 0$$

we thus obtain Lamé's formula †

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ = \frac{1}{\lambda \mu \nu} \left[\frac{\partial}{\partial \alpha} \left(\frac{\mu \nu}{\lambda} \frac{\partial V}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{\nu \lambda}{\mu} \frac{\partial V}{\partial \beta} \right) + \frac{\partial}{\partial \gamma} \left(\frac{\lambda \mu}{\nu} \frac{\partial V}{\partial \gamma} \right) \right] \quad (12) \end{aligned}$$

Let us first of all transform Laplace's equation to cylindrical coordinates (ρ, φ, z) , where $x = \rho \cos \varphi$, $y = \rho \sin \varphi$; we have then

$$dx \delta x + dy \delta y + dz \delta z = d\rho \delta \rho + \rho^2 d\varphi \delta \varphi + dz \delta z \quad (13)$$

hence the new form of Laplace's equation is

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 V}{\partial \varphi^2} + \rho \frac{\partial^2 V}{\partial z^2} = 0 \quad (14)$$

This equation is satisfied by $V = u(\rho) v(\varphi) w(z)$ if

$$uvw \left[\frac{1}{u} \frac{d}{d\rho} \left(\rho \frac{du}{d\rho} \right) + \frac{1}{\rho v} \frac{d^2 v}{d\varphi^2} + \frac{\rho}{w} \frac{d^2 w}{dz^2} \right] = 0$$

This requires that $\frac{\partial^2 v}{d\varphi^2} + m^2 v = 0$

$$\frac{d^2 w}{dz^2} - k^2 w = 0$$

and

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{du}{d\rho} \right) - \left(\frac{m^2}{\rho^2} - k^2 \right) u = 0$$

where m and k are constants.

* This result is obtained in another manner by J. Larmor, *Camb. Phil. Trans.*, Vol. 14 (1885), p. 121.

† This result is usually obtained by the geometrical method due to Lord Kelvin. See Lamb's *Hydrodynamics*, p. 141.

Hence there are particular solutions of type

$$V = e^{\pm k z} \cos(m\varphi + \varepsilon) F_m(k\rho)$$

where ε is a constant and $F_m(s)$ is a solution of *Bessel's equation* *

$$\frac{d^2 u}{ds^2} + \frac{1}{s} \frac{du}{ds} + \left(1 - \frac{m^2}{s^2}\right) u = 0$$

2 Let us next transform Laplace's equation to polar coordinates (r, θ, φ) , where

$$x = r \sin \theta \cos \varphi = \rho \cos \varphi \quad y = r \sin \theta \sin \varphi = \rho \sin \varphi \quad z = r \cos \theta$$

In this case we have

$$dx \delta x + dy \delta y + dz \delta z = dr \delta r + r^2 d\theta \delta \theta + r^2 \sin^2 \theta d\varphi \delta \varphi \quad (15)$$

Hence the new form of Laplace's equation is

$$\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \operatorname{cosec} \theta \frac{\partial^2 V}{\partial \varphi^2} = 0 \quad (16)$$

This equation is satisfied by $V = u(r) v(\theta) w(\varphi)$ if

$$uvw \left[\frac{1}{u} \sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{v} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) + \frac{\operatorname{cosec} \theta}{w} \frac{\partial^2 w}{\partial \varphi^2} \right] = 0$$

Writing
$$\frac{d^2 w}{d\varphi^2} + m^2 w = 0$$

$$\frac{d}{dr} \left(r^2 \frac{du}{dr} \right) = n(n+1)u$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dv}{d\theta} \right) + \left[n(n+1) - \frac{m^2}{\sin^2 \theta} \right] v = 0$$

where m and n are constants, the above equations are satisfied.

If we write $\cos \theta = \sigma$, the last equation becomes

$$\frac{d}{d\sigma} \left[(1 - \sigma^2) \frac{dv}{d\sigma} \right] + \left[n(n+1) - \frac{m^2}{1 - \sigma^2} \right] v = 0 \quad (17)$$

When $m = 0$ this reduces to Legendre's equation. The equation for u is satisfied by $u = r^n$ and $u = r^{-(n+1)}$; hence we have particular solutions of Laplace's equation of the types

$$\begin{aligned} & r^n \cos(m\varphi + \varepsilon) P_n^m(\sigma) \\ & r^{-n-1} \cos(m\varphi + \varepsilon) P_n^m(\sigma) \end{aligned}$$

where $v = P_n^m(\sigma)$ is a solution * of equation (17).

A comparison of these two solutions suggests the following theorem:

If $V = F(r, \theta, \varphi)$ is a solution of Laplace's equation, then the function $V = \frac{1}{r} F\left(\frac{1}{r}, \theta, \varphi\right)$ is also a solution of Laplace's equation.

* This equation will be solved by the method of infinite series in Chapter IX.

This is easily proved ; for if we write $r = \frac{1}{s}$ in equation (16), it becomes

$$\sin \theta \left(s^2 \frac{\partial^2 V}{\partial s^2} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \operatorname{cosec} \theta \frac{\partial^2 V}{\partial \varphi^2} = 0$$

A further substitution $V = sU$ reduces this to the form

$$\sin \theta \left(s^2 \frac{\partial^2 U}{\partial s^2} + 2s \frac{\partial U}{\partial s} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \operatorname{cosec} \theta \frac{\partial^2 U}{\partial \varphi^2} = 0$$

which is of the same form as (16), but with s , U written in place of r , V respectively. Hence the theorem is proved.

The preceding theorem enables us to derive one solution of Laplace's equation from another by means of the transformation of coordinates

$$x' = \frac{x}{r^2} \quad y' = \frac{y}{r^2} \quad z' = \frac{z}{r^2}$$

This transformation is known as *inversion*. It was applied with great success to electrostatic problems by Lord Kelvin. The preceding theorem may also be enunciated as follows :

If $F(x, y, z)$ is a solution of Laplace's equation, then

$$\frac{1}{r} F \left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right)$$

is also a solution of Laplace's equation.

All the preceding results may be extended to the case of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial s^2} = 0 \quad \dots \quad (18)$$

which is derived from the equation of wave propagation by putting $s = ict$. If we write

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = \sigma \cos \psi \quad s = \sigma \sin \psi \quad (19)$$

we have
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2}$$

as in the transformation of Laplace's equation to cylindrical coordinates ; also

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial s^2} = \frac{1}{\sigma} \frac{\partial}{\partial \sigma} \left(\sigma \frac{\partial V}{\partial \sigma} \right) + \frac{1}{\sigma^2} \frac{\partial^2 V}{\partial \psi^2}$$

Hence equation (18) becomes

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial \sigma^2} + \frac{1}{\sigma} \frac{\partial V}{\partial \sigma} + \frac{1}{\sigma^2} \frac{\partial^2 V}{\partial \psi^2} = 0 \quad (20)$$

This is satisfied by

$$V = \cos (m\varphi + \varepsilon) \cos (p\psi + \eta) F(\rho) G(\sigma)$$

where m, p, ϵ, η are constants if

$$\frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) F = 0$$

$$\frac{d^2 G}{d\sigma^2} + \frac{1}{\sigma} \frac{dG}{d\sigma} - \left(k^2 + \frac{p^2}{\sigma^2}\right) G = 0$$

where k is a constant. If we put $k\rho = \xi$, $ik\sigma = \zeta$, the two last equations are transformed into cases of Bessel's equation.

If, on the other hand, we make the further substitution

$$\rho = \tau \sin \theta, \quad \sigma = \tau \cos \theta$$

we have

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{\partial^2 V}{\partial \sigma^2} = \frac{1}{\tau} \frac{\partial}{\partial \tau} \left(\tau \frac{\partial V}{\partial \tau} \right) + \frac{1}{\tau^2} \frac{\partial^2 V}{\partial \theta^2}$$

$$\frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\sigma} \frac{\partial V}{\partial \sigma} = \frac{2}{\tau} \frac{\partial V}{\partial \tau} + \frac{1}{\tau^2} (\cot \theta - \tan \theta) \frac{\partial V}{\partial \theta}$$

Hence equation (20) becomes

$$\frac{\partial^2 V}{\partial \tau^2} + \frac{3}{\tau} \frac{\partial V}{\partial \tau} + \frac{1}{\tau^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{2 \cot 2\theta}{\tau^2} \frac{\partial V}{\partial \theta}$$

$$+ \frac{1}{\tau^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} + \frac{1}{\tau^2 \cos^2 \theta} \frac{\partial^2 V}{\partial \psi^2} = 0$$

This is satisfied by $V = \tau^{2n} \cos(m\varphi + \epsilon) \cos(p\psi + \eta) F(\theta)$ if F satisfies the differential equation

$$\frac{d^2 F}{d\theta^2} + 2 \cot 2\theta \frac{dF}{d\theta} + \left(4n(n+1) - \frac{m^2}{\sin^2 \theta} - \frac{p^2}{\cos^2 \theta}\right) F = 0 \quad (21)$$

This equation is unaltered when $-(n+1)$ is written in place of n ; hence a second solution is given by

$$V = \tau^{-2n-2} \cos(m\varphi + \epsilon) \cos(p\psi + \eta) F(\theta)$$

This suggests the following theorem :

If $V = f(\tau, \theta, \varphi, \psi)$ is a solution of equation (18), then

$$V = \frac{1}{\tau^2} f\left(\frac{1}{\tau}, \theta, \varphi, \psi\right)$$

is also a solution.

This is easily verified by a slight extension of the method used in the case of the corresponding theorem for Laplace's equation. The theorem can also be enunciated as follows :

If $V = f(x, y, z, s)$ is a solution of equation (18), then

$$V = \frac{1}{\tau^2} f\left(\frac{x}{\tau^2}, \frac{y}{\tau^2}, \frac{z}{\tau^2}, \frac{s}{\tau^2}\right)$$

is also a solution, where

$$\tau^2 = x^2 + y^2 + z^2 + s^2$$

The transformation

$$x' = \frac{x}{\tau^2} \quad y' = \frac{y}{\tau^2} \quad z' = \frac{z}{\tau^2} \quad s' = \frac{s}{\tau^2}$$

is called *generalised inversion*.

. If in equation (21) we write $p^2 = m^2$, $\cos 2\theta = \mu$, it becomes

$$\frac{d}{d\mu} \left[(1 - \mu^2) \frac{dF}{d\mu} \right] + \left[n(n+1) - \frac{m^2}{1 - \mu^2} \right] F = 0$$

We thus have solutions of equation (18) of the form

$$V = \tau^{2n} P_n^m(\mu) \cos(m\varphi + \varepsilon) \cos(m\psi \pm \eta)$$

and since ε and η are arbitrary constants, we may deduce that

$$\tau^{2n} P_n^m(\mu) \cos m(\varphi \pm \psi) \quad \text{and} \quad \tau^{2n} P_n^m(\mu) \sin m(\varphi \pm \psi)$$

are solutions of equation (18). Comparing these with the corresponding solutions of Laplace's equation, viz.

$$r^n P_n^m(\sigma) \cos m\varphi \quad r^n P_n^m(\sigma) \sin m\varphi$$

the following theorem is suggested :

If $V = f(r, \theta, \varphi)$ is a solution of Laplace's equation, the function $V = f(\tau^2, 2\theta, \varphi \pm \psi)$ is a solution of equation (18).

This may be verified by a direct comparison of the differential equations, to be satisfied by the function f in the two cases.

In the general case when $p^2 \neq m^2$, the substitution $\sin^2 \theta = \xi$ reduces equation (21) to the form

$$\xi(1 - \xi) \frac{d^2 F}{d\xi^2} + (1 - 2\xi) \frac{dF}{d\xi} + \left[n(n+1) - \frac{m^2}{4\xi} - \frac{p^2}{4(1 - \xi)} \right] F = 0$$

Putting $F = \xi^{\frac{m}{2}} (1 - \xi)^{\frac{p}{2}} G$, we obtain the hypergeometric equation

$$\xi(1 - \xi) \frac{d^2 G}{d\xi^2} + [\gamma - (\alpha + \beta + 1)\xi] \frac{dG}{d\xi} - \alpha\beta G = 0$$

where

$$\gamma = m + 1 \quad \alpha = n + 1 + \frac{1}{2}(m + p) \quad \beta = \frac{1}{2}(m + p) - n$$

Hence, if $F(\alpha, \beta, \gamma, \xi)$ denote a solution of the hypergeometric equation, we have solutions of equation (18) of type

$$V = \tau^{2n} \cos(m\varphi + \varepsilon) \cos(p\psi + \eta) \sin^m \theta \cos^p \theta F(\alpha, \beta, \gamma, \sin^2 \theta)$$

This result is due to Stieltjes.

Let us next consider the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 \quad . \quad . \quad . \quad (22)$$

which is derived from the equation of wave propagation by assuming that V is of the form

$$V = F(x, y, z) \cos kc(t - t_0)$$

where k and t_0 are constants, and F is a function to be determined.

Transforming to polar coordinates, we obtain

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + k^2 r^2 V + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} = 0 \quad (23)$$

This is satisfied by $V = u(r) v(\theta) w(\varphi)$ if

$$\frac{d^2 w}{d\varphi^2} + m^2 w = 0$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dv}{d\theta} \right) + \left[n(n+1) - \frac{m^2}{\sin^2 \theta} \right] v = 0$$

$$\frac{d}{dr} \left(r^2 \frac{du}{dr} \right) + k^2 r^2 u = n(n+1)u$$

Putting $kr = s$, $u = s^{-\frac{1}{2}}U$, we find that U is a solution of Bessel's equation

$$\frac{d^2 U}{ds^2} + \frac{1}{s} \frac{dU}{ds} + \left(1 - \frac{(n + \frac{1}{2})^2}{s^2} \right) U = 0. \quad (24)$$

If $U = J_{n+\frac{1}{2}}(s)$ is a solution of this equation, we have solutions of the original equation (22) of type

$$V = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr) \cos(m\varphi + \varepsilon) P_n^m(\cos \theta)$$

where n , m , k , ε are arbitrary constants. The solutions in which n and m are integers are of chief interest; a solution of (24) may then be deduced from the results of § 44.

There are many other transformations of coordinates which enable us to construct potential functions of type $V = u(\alpha) v(\beta) w(\gamma)$, but an exhaustive discussion of these is beyond the scope of the present work. We shall now pass on to the problem of finding potential functions of the form $V = \gamma f(\theta)$, where γ and θ are certain functions of x , y , z and $f(\theta)$ is an arbitrary function of θ .

If we substitute this expression in the differential equation

$$\Delta V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

we get

$$\begin{aligned} f(\theta) \Delta \gamma + f'(\theta) \left[\gamma \Delta \theta + 2 \left(\frac{\partial \gamma}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \gamma}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial \gamma}{\partial z} \frac{\partial \theta}{\partial z} \right) \right] \\ + \gamma f''(\theta) \left[\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 + \left(\frac{\partial \theta}{\partial z} \right)^2 \right] = 0 \end{aligned}$$

Since $f(\theta)$ is arbitrary, we must have $\Delta \gamma = 0$, and

$$\gamma \Delta \theta + 2 \left(\frac{\partial \gamma}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \gamma}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial \gamma}{\partial z} \frac{\partial \theta}{\partial z} \right) = 0 \quad (25)$$

$$\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 + \left(\frac{\partial \theta}{\partial z} \right)^2 = 0 \quad (26)$$

If $\frac{\partial \theta}{\partial x}$ is not zero, we may evidently write

$$\mu \frac{\partial \theta}{\partial x} = -i \quad \mu \frac{\partial \theta}{\partial y} = \cos \alpha \quad \mu \frac{\partial \theta}{\partial z} = \sin \alpha \quad (27)$$

where μ and α are functions of x, y, z . We shall now show that these functions satisfy the equations

$$\frac{\partial \mu}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \mu}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial \mu}{\partial z} \frac{\partial \theta}{\partial z} = 0 \quad . \quad . \quad . \quad (28)$$

$$\frac{\partial \alpha}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial \alpha}{\partial z} \frac{\partial \theta}{\partial z} = 0 \quad . \quad . \quad . \quad (29)$$

If, in fact, we differentiate the first of equations (27), we obtain

$$\left. \begin{aligned} \frac{\partial \mu}{\partial x} \frac{\partial \theta}{\partial x} + \mu \frac{\partial^2 \theta}{\partial x^2} &= 0 & \frac{\partial \mu}{\partial y} \frac{\partial \theta}{\partial x} + \mu \frac{\partial^2 \theta}{\partial x \partial y} &= 0 \\ \frac{\partial \mu}{\partial z} \frac{\partial \theta}{\partial x} + \mu \frac{\partial^2 \theta}{\partial x \partial z} &= 0 \end{aligned} \right\} \quad (30)$$

while if we differentiate (26) with regard to x , we obtain

$$\frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial y} \frac{\partial^2 \theta}{\partial x \partial y} + \frac{\partial \theta}{\partial z} \frac{\partial^2 \theta}{\partial x \partial z} = 0 \quad . \quad . \quad . \quad (31)$$

Equation (28) is now seen to be a consequence of (30) and (31).

Again, if we differentiate the third of equations (27), we get

$$\left. \begin{aligned} \frac{\partial \mu}{\partial x} \frac{\partial \theta}{\partial z} + \mu \frac{\partial^2 \theta}{\partial x \partial z} &= \cos \alpha \frac{\partial \alpha}{\partial x} & \frac{\partial \mu}{\partial y} \frac{\partial \theta}{\partial z} + \mu \frac{\partial^2 \theta}{\partial y \partial z} &= \cos \alpha \frac{\partial \alpha}{\partial y} \\ \frac{\partial \mu}{\partial z} \frac{\partial \theta}{\partial z} + \mu \frac{\partial^2 \theta}{\partial z^2} &= \cos \alpha \frac{\partial \alpha}{\partial z} = 0 \end{aligned} \right\} \quad (32)$$

while if we differentiate (26) with regard to z , we get

$$\frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x \partial z} + \frac{\partial \theta}{\partial y} \frac{\partial^2 \theta}{\partial y \partial z} + \frac{\partial \theta}{\partial z} \frac{\partial^2 \theta}{\partial z^2} = 0 \quad . \quad . \quad . \quad (33)$$

Equation (29) is now seen to be a consequence of (32) and (33), provided $\cos \alpha \neq 0$. Let us now write

$$v = y - ix \cos \alpha \quad w = z - ix \sin \alpha \quad . \quad . \quad (34)$$

then $\mu \frac{\partial(\theta, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} -i & \cos \alpha & \sin \alpha \\ -i \cos \alpha + ix \sin \alpha \frac{\partial \alpha}{\partial x} & 1 + ix \sin \alpha \frac{\partial \alpha}{\partial y} & ix \sin \alpha \frac{\partial \alpha}{\partial z} \\ -i \sin \alpha - ix \cos \alpha \frac{\partial \alpha}{\partial x} & -ix \cos \alpha \frac{\partial \alpha}{\partial y} & 1 - ix \cos \alpha \frac{\partial \alpha}{\partial z} \end{vmatrix}$$

and the determinant is seen to vanish.

Similarly, it can be shown that

$$\frac{\partial(\alpha, v, w)}{\partial(x, y, z)} = 0$$

in virtue of (27) and (29). We must conclude then that there is either a relation between v and w , or that there are relations expressing α and θ as functions of v and w . Now

$$\mu d\theta = dy \cos \alpha + dz \sin \alpha - i dx = dv \cos \alpha + dw \sin \alpha$$

hence we have the following general method of finding a function θ which satisfies equation (26) and is not independent of either x or y .

Let v and w be defined by the equations (34), wherein α, v, w are connected by a relation not involving x, y, z ; then, if θ be defined by the equation

$$dv \cos \alpha + dw \sin \alpha = \mu d\theta$$

it is a solution of (26). Conversely, any solution of (26) which involves x and y can be obtained in this way.

If θ is a function of α , we have

$$d(v \cos \alpha + w \sin \alpha) = \mu d\theta + (w \cos \alpha - v \sin \alpha) d\alpha = \lambda d\alpha, \text{ say}$$

hence $v \cos \alpha + w \sin \alpha$ is a function of α , and there is a relation of the form

$$y \cos \alpha + z \sin \alpha - ix = \varphi(\alpha) \quad . \quad . \quad . \quad (35)$$

This brings us to a case which has been discussed by Jacobi, Forsyth, and Levi Civita. We have

$$M \frac{\partial \alpha}{\partial x} = -i \quad M \frac{\partial \alpha}{\partial y} = \cos \alpha \quad M \frac{\partial \alpha}{\partial z} = \sin \alpha$$

where

$$M = \varphi'(\alpha) - z \cos \alpha + y \sin \alpha \quad . \quad . \quad . \quad (36)$$

Again,

$$M \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial M}{\partial \alpha} \left(\frac{\partial \alpha}{\partial x} \right)^2 = 0$$

$$M \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial M}{\partial \alpha} \left(\frac{\partial \alpha}{\partial y} \right)^2 + \sin \alpha \frac{\partial \alpha}{\partial y} = -\sin \alpha \frac{\partial \alpha}{\partial y}$$

$$M \frac{\partial^2 \alpha}{\partial z^2} + \frac{\partial M}{\partial \alpha} \left(\frac{\partial \alpha}{\partial z} \right)^2 - \cos \alpha \frac{\partial \alpha}{\partial z} = \cos \alpha \frac{\partial \alpha}{\partial z}$$

Adding up, and remembering that

$$\left(\frac{\partial \alpha}{\partial x} \right)^2 + \left(\frac{\partial \alpha}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial z} \right)^2 = 0$$

we have

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = 0$$

Hence the conditions (25) and (26) are satisfied by $\gamma = 1, \alpha = \theta$, and we have the result that $V = f(\alpha)$ is a solution of Laplace's equation, f being an arbitrary function.

Since $\frac{\partial V}{\partial x}$ is also a solution of Laplace's equation and $f'(\alpha)$ can also be regarded as an arbitrary function, we have the result that

$$W = \frac{\partial V}{\partial x} = -\frac{i}{M} f'(\alpha) = \frac{g(\alpha)}{M}$$

is a solution of Laplace's equation, g being an arbitrary function.

We conclude, then, that if $\theta = \alpha$, where α is defined by (35), we may take

$$\gamma = a(\alpha) + \frac{b(\alpha)}{M}$$

where $a(\alpha)$ and $b(\alpha)$ are arbitrary functions of α .

We shall show presently that this is the most general value of γ .

Let us next consider the case in which α is a constant; θ is then a function of $y \cos \alpha + z \sin \alpha - ix$, and it is easy to verify that $V = f(\theta)$ is a solution of Laplace's equation. A second solution is derived from this by differentiating with regard to the parameter α ; thus a second solution is

$$\frac{\partial V}{\partial \alpha} = (y \cos \alpha - y \sin \alpha) f'(\theta) (y \cos \alpha + z \sin \alpha - ix)$$

When θ has the value $y \cos \alpha + z \sin \alpha - ix$, an appropriate value of γ is consequently

$$\gamma = a(\theta) + b(\theta) (z \cos \alpha - y \sin \alpha)$$

where $a(\theta)$ and $b(\theta)$ are arbitrary functions of θ . It will be shown presently that this is the most general value of γ .

The case in which θ is independent of x or y leads to a class of potential functions which may be derived from the previous one by a simple interchange of variables and a special choice of α . We have, in fact, $\theta = y + iz$, and it is easy to see that in this case $V = \gamma f(\theta)$ satisfies Laplace's equation if

$$\gamma = a(\theta) + xb(\theta)$$

where $a(\theta)$ and $b(\theta)$ are arbitrary functions.

Let us finally consider the case in which θ depends on x and is not simply a function of α ; we may then write equations (34) in the form

$$y = ix \cos \alpha + f(\alpha, \theta) \quad z = ix \sin \alpha + g(\alpha, \theta) \quad (37)$$

$$\text{The equation} \quad df \cos \alpha + dg \sin \alpha = \mu d\theta \quad . \quad . \quad . \quad (38)$$

$$\text{implies that} \quad \frac{\partial f}{\partial \alpha} \cos \alpha + \frac{\partial g}{\partial \alpha} \sin \alpha = 0$$

and so we may write

$$\frac{\partial f}{\partial \alpha} = i\xi \sin \alpha \quad \frac{\partial g}{\partial \alpha} = -i\xi \cos \alpha \quad . \quad . \quad . \quad (39)$$

We now require the new form of Laplace's equation when x, α, θ are taken as independent variables. Let us write

$$\frac{\partial f}{\partial \theta} = i\sigma \cos \epsilon \quad \frac{\partial g}{\partial \theta} = i\sigma \sin \epsilon$$

then it is easily seen that

$$\begin{aligned} dx \delta x + dy \delta y + dz \delta z \\ = - (x - \xi)^2 d\alpha \delta \alpha - \sigma^2 d\theta \delta \theta - \sigma \cos (\alpha - \epsilon) (dx \delta \theta + \delta x d\theta) \\ + \sigma (x - \xi) \sin (\alpha - \epsilon) (d\alpha \delta \theta + \delta \alpha d\theta) \end{aligned}$$

The determinant of the coefficients is in this case

$$\Delta = \begin{vmatrix} - (x - \xi)^2 & \sigma (x - \xi) \sin (\alpha - \epsilon) & 0 \\ \sigma (x - \xi) \sin (\alpha - \epsilon) & - \sigma^2 & - \sigma \cos (\alpha - \epsilon) \\ 0 & - \sigma \cos (\alpha - \epsilon) & 0 \end{vmatrix} \\ = + \sigma^2 (x - \xi)^2 \cos^2 (\alpha - \epsilon)$$

The new form of Laplace's equation is consequently

$$\left. \begin{aligned} \frac{\partial}{\partial \alpha} \left[\frac{\sigma}{x - \xi} \cos (\alpha - \epsilon) \frac{\partial V}{\partial \alpha} + \sigma \sin (\alpha - \epsilon) \frac{\partial V}{\partial x} \right] \\ + \frac{\partial}{\partial \theta} \left[(x - \xi) \frac{\partial V}{\partial x} \right] + \frac{\partial}{\partial x} \left[\sigma \sin (\alpha - \epsilon) \frac{\partial V}{\partial \alpha} + (x - \xi) \frac{\partial V}{\partial \theta} \right] \\ - \sigma (x - \xi) \cos (\alpha - \epsilon) \frac{\partial V}{\partial x} \end{aligned} \right\} = 0 \quad (40)$$

Let us now substitute $V = \gamma f(\theta)$ in this equation and equate to zero the coefficient of $f'(\theta)$. The resulting equation is

$$(x - \xi) \frac{\partial \gamma}{\partial x} + \frac{\partial}{\partial x} [\gamma (x - \xi)] = 0$$

and this implies that $\gamma^2 (x - \xi)$ depends only on α and θ . Hence we may put

$$V = \gamma f(\theta) = (x - \xi)^{-\frac{1}{2}} \Phi(\alpha, \theta)$$

When this last expression for V is substituted in the differential equation (40), it is found that there is one term with $(x - \xi)^{-\frac{1}{2}}$ as a factor and no other term which could possibly cancel it. Equating this term to zero, we find that we must have

$$\sigma \cos (\alpha - \epsilon) \Phi(\alpha, \theta) \left(\frac{\partial \xi}{\partial \alpha} \right)^2 = 0 \quad \dots \quad (41)$$

$$\text{Now} \quad \sigma \xi \cos (\alpha - \epsilon) = \frac{\partial (f, g)}{\partial (\alpha, \theta)}$$

consequently, if $\sigma \cos (\alpha - \epsilon)$ is zero, there is a relation between f and g , and the equation (38) then indicates that both f and g are functions of α . This implies the existence of a relation between

x , y , and z ; consequently the possibility of $\sigma \cos(\alpha - \epsilon)$ being zero may be dismissed.

We must conclude then, from equation (41), that $\frac{\partial \xi}{\partial \alpha} = 0$ or that ξ is a function of θ only. Integrating equations (39), we now find that

$$f = -i\xi \cos \alpha + \eta(\theta) \qquad g = -i\xi \sin \alpha + \zeta(\theta)$$

hence

$$y - \eta(\theta) = i \cos \alpha [x - \xi(\theta)] \qquad z - \zeta(\theta) = i \sin \alpha [x - \xi(\theta)]$$

Eliminating α , we find that θ must be defined in terms of x , y , z by an equation of type

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0. \quad (42)$$

An appropriate value of γ will be found in Chapter X, but the result will be stated here.

Let l, m, n ; l_1, m_1, n_1 be two independent sets of three functions of θ which satisfy relations of type

$$l^2 + m^2 + n^2 = 0$$

$$l \frac{d\xi}{d\theta} + m \frac{d\eta}{d\theta} + n \frac{d\zeta}{d\theta} = 0$$

and let

$$w = l(x - \xi) + m(y - \eta) + n(z - \zeta)$$

$$w_1 = l_1(x - \xi) + m_1(y - \eta) + n_1(z - \zeta)$$

then $V = w^{-\frac{1}{2}}f(\theta)$ and $V = w_1^{-\frac{1}{2}}f(\theta)$ are solutions of Laplace's equation. The general value of γ is consequently

$$\gamma = w^{-\frac{1}{2}}a(\theta) + w_1^{-\frac{1}{2}}b(\theta)$$

where $a(\theta)$ and $b(\theta)$ are arbitrary functions of θ .

We shall now prove that when θ satisfies the conditions that potential functions of type $V = \gamma f(\theta)$ may exist, the most general value of γ is

$$\gamma = \gamma_1 a(\theta) + \gamma_2 b(\theta)$$

where γ_1 and γ_2 are two particular values of γ whose ratio is not simply a function of θ .

To do this we take $\gamma_1, \gamma_2, \theta$ as new independent variables, and assume that Laplace's equation is transformed into the equation

$$\begin{aligned} A \frac{\partial^2 V}{\partial \gamma_1^2} + 2H \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} + B \frac{\partial^2 V}{\partial \gamma_2^2} + C \frac{\partial^2 V}{\partial \theta^2} + 2F \frac{\partial^2 V}{\partial \theta \partial \gamma_1} + 2G \frac{\partial^2 V}{\partial \theta \partial \gamma_2} \\ + 2P \frac{\partial V}{\partial \gamma_1} + 2Q \frac{\partial V}{\partial \gamma_2} + 2R \frac{\partial V}{\partial \theta} + SV = 0 \end{aligned}$$

Since this is satisfied by $V = \gamma_1 f(\theta)$ and $V = \gamma_2 f(\theta)$, we must have

$$\begin{aligned} C = 0 \qquad 2F + 2R\gamma_1 = 0 \qquad 2G + 2R\gamma_2 = 0 \\ 2P + S\gamma_1 = 0 \qquad 2Q + S\gamma_2 = 0 \end{aligned}$$

The differential equation may consequently be written in the form

$$\left. \begin{aligned} & A \frac{\partial^2 V}{\partial \gamma_1^2} + 2H \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} + B \frac{\partial^2 V}{\partial \gamma_2^2} \\ & + 2R \left[\frac{\partial V}{\partial \theta} - \gamma_1 \frac{\partial^2 V}{\partial \gamma_1 \partial \theta} - \gamma_2 \frac{\partial^2 V}{\partial \gamma_2 \partial \theta} \right] \\ & + S \left[V - \gamma_1 \frac{\partial V}{\partial \gamma_1} - \gamma_2 \frac{\partial V}{\partial \gamma_2} \right] = 0 \end{aligned} \right\} \quad (43)$$

In order that this equation may be satisfied by $V = \gamma f(\theta)$, where f is an arbitrary function, we must have

$$\begin{aligned} \gamma_1 \frac{\partial V}{\partial \gamma_1} + \gamma_2 \frac{\partial V}{\partial \gamma_2} &= V \\ A \frac{\partial^2 V}{\partial \gamma_1^2} + 2H \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} + B \frac{\partial^2 V}{\partial \gamma_2^2} &= 0 \end{aligned}$$

Differentiating the first of these equations, we get

$$\gamma_1 \frac{\partial^2 V}{\partial \gamma_1^2} + \gamma_2 \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} = 0 \quad \gamma_1 \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} + \gamma_2 \frac{\partial^2 V}{\partial \gamma_2^2} = 0$$

hence, either

$$\frac{\partial^2 V}{\partial \gamma_1^2} = 0 \quad \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2} = 0 \quad \frac{\partial^2 V}{\partial \gamma_2^2} = 0$$

or

$$A\gamma_2^2 - 2H\gamma_1\gamma_2 + B\gamma_1^2 = 0$$

In the first case $V = [\gamma_1 a(\theta) + \gamma_2 b(\theta)] f(\theta)$. In the second case equation (43) can be written in the form

$$A \frac{1}{\gamma_1} \frac{\partial W}{\partial \gamma_1} + B \frac{1}{\gamma_2} \frac{\partial W}{\partial \gamma_2} - 2R \frac{\partial W}{\partial \theta} - SW = 0$$

where

$$W = \gamma_1 \frac{\partial V}{\partial \gamma_1} + \gamma_2 \frac{\partial V}{\partial \gamma_2} - V$$

This, however, is not possible; for if Laplace's equation could be thrown into this form, it would be satisfied whenever $W = 0$, that is, when

$$V = \gamma_1 F\left(\frac{\gamma_2}{\gamma_1}, \theta\right)$$

F being an arbitrary function. Now, by hypothesis, $\frac{\gamma_2}{\gamma_1}$ is not

simply a function of θ ; consequently, if we write $\varphi = \frac{\gamma_2}{\gamma_1}$ and substitute the above expression for V in the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

we obtain

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 = 0$$

$$\frac{\partial \theta}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial \theta}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial z} \frac{\partial \varphi}{\partial z} = 0$$

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 = 0$$

and these equations give, on eliminating $\frac{\partial \varphi}{\partial x}, \frac{\partial \theta}{\partial x},$

$$\left(\frac{\partial \theta}{\partial y} \frac{\partial \varphi}{\partial z} - \frac{\partial \theta}{\partial z} \frac{\partial \varphi}{\partial y}\right)^2 = 0 \quad \text{or} \quad \frac{\frac{\partial \theta}{\partial y}}{\frac{\partial \varphi}{\partial y}} = \frac{\frac{\partial \theta}{\partial z}}{\frac{\partial \varphi}{\partial z}}$$

This implies that

$$\frac{\frac{\partial \theta}{\partial x}}{\frac{\partial \varphi}{\partial x}} = \frac{\frac{\partial \theta}{\partial y}}{\frac{\partial \varphi}{\partial y}} = \frac{\frac{\partial \theta}{\partial z}}{\frac{\partial \varphi}{\partial z}}$$

hence φ must be a function of θ , which is contrary to hypothesis.

The only possibility, then, is that the second case is inadmissible, and so we must have

$$\gamma = \gamma_1 a(\theta) + \gamma_2 b(\theta)$$

The preceding results may be collected together in the following theorem :

In order that Laplace's equation may be satisfied by an expression of type $V = \gamma f(\theta)$, in which the function f is arbitrary, the quantity θ must either be defined by an equation of the form

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0$$

or by an equation of the form

$$xl(\theta) + ym(\theta) + zn(\theta) = p(\theta)$$

where l, m, n are either constants or functions of θ connected by the relation

$$l^2 + m^2 + n^2 = 0$$

The most general value of γ is in each case of the form

$$\gamma = \gamma_1 a(\theta) + \gamma_2 b(\theta)$$

where γ_1 and γ_2 are particular values of γ , whose ratio is not simply a function of θ . In the first case we may take $\gamma_1 = w^{-\frac{1}{2}}, \gamma_2 = w_1^{-\frac{1}{2}}$, where w and w_1 are the functions defined above. In the second case we may take $\gamma_1 = 1$ and define γ_2 by the equation

$$\frac{1}{\gamma_2} = x \frac{dl_1}{d\theta} + y \frac{dm}{d\theta} + z \frac{dn}{d\theta} - \frac{dp}{d\theta}$$

EXAMPLES.

1. Prove that Laplace's equation is satisfied by

$$V = f\left(\frac{x+iy}{z+r}\right) \quad \text{and} \quad V = \frac{1}{r} f\left(\frac{x+iy}{z+r}\right)$$

where $r^2 = x^2 + y^2 + z^2$, and f is an arbitrary function. (Donkin.)

2. Prove that the function

$$V = \frac{1}{\sqrt{x+iy}} f(z \pm i\rho)$$

satisfies Laplace's equation, where $\rho^2 = x^2 + y^2$, and f is an arbitrary function. (Poisson.)

3. Prove that the following functions satisfy Laplace's equation :

$$\log \rho \quad \tan^{-1} \frac{y}{x} \quad \log \frac{r+z}{r-z} \quad \frac{1}{r} \log \frac{r+z}{r-z} \quad \tan^{-1} \frac{y}{x} \cdot \log \frac{r+z}{r-z}$$

4. Prove that if
- $x = \rho \cos \phi$
- ,
- $y = \rho \sin \phi$
- ,
- $z = \sigma - m\phi$
- ,
- m
- being a constant, Laplace's equation is satisfied by a ' helicoidal potential '
- $V = F(\rho, \sigma)$
- , where
- F
- satisfies the partial differential equation

$$\frac{\partial^2 F}{\partial \rho^2} + \left(1 + \frac{m^2}{\rho^2}\right) \frac{\partial^2 F}{\partial \sigma^2} + \frac{1}{\rho} \frac{\partial F}{\partial \rho} = 0$$

(T. Levi Civita.)

5. Prove that if

$$x = \alpha \sin \beta e^{m\phi} \cos \phi \quad y = \alpha \sin \beta e^{m\phi} \sin \phi \quad z = \alpha \cos \beta e^{m\phi}$$

Laplace's equation is satisfied by a ' spiral potential ' $V = F(\alpha, \beta)$, where F satisfies the partial differential equation

$$\left(1 + \frac{m^2}{\sin^2 \beta}\right) \frac{\partial^2 F}{\partial \alpha^2} + \frac{1}{\alpha^2} \frac{\partial^2 F}{\partial \beta^2} + \frac{1}{\alpha} \left(2 + \frac{m^2}{\sin^2 \beta}\right) \frac{\partial F}{\partial \alpha} + \frac{1}{\alpha^2} \cot \beta \frac{\partial F}{\partial \beta} = 0$$

and m is a constant. (T. Levi Civita.)

6. Let
- α, β, ω
- be defined in terms of
- x, y, z
- by the equations

$$\gamma^2 x = \cos \alpha \cos (\beta + k\omega) \quad \gamma^2 y = \cos \alpha \sin (\beta + k\omega) \quad \gamma^2 z = \sin \alpha \cos \omega$$

where $\gamma^2 = 1 - \sin \alpha \sin \omega$ and k is a constant. A solution of Laplace's equation is then given by $V = \gamma F(\alpha, \beta)$, provided F satisfies the partial differential equation

$$\sin 2\alpha \frac{\partial^2 F}{\partial \alpha^2} + 4 \frac{\sin^2 \alpha + k^2 \cos^2 \alpha}{\sin 2\alpha} \frac{\partial^2 F}{\partial \beta^2} + 2 \cos 2\alpha \frac{\partial F}{\partial \alpha} - \frac{3}{4} \sin 2\alpha \cdot F = 0$$

(V. Amaldi.)

7. Prove that if we put

$$x = \rho \cos \phi \quad y = \rho \sin \phi \quad \rho = aM \sinh \sigma \quad z = aM \sin \psi$$

where a is a constant and $M^{-1} = \cosh \sigma - \cos \psi$, Laplace's equation becomes

$$\frac{\partial}{\partial \sigma} \left(M \sinh \sigma \frac{\partial u}{\partial \sigma} \right) + \frac{\partial}{\partial \psi} \left(M \sinh \sigma \frac{\partial u}{\partial \psi} \right) + M \operatorname{cosech} \sigma \frac{\partial^2 u}{\partial \phi^2} = 0$$

Show also that this equation is satisfied by

$$u = M^{-1} \cos n(\psi - \psi_0) \cos m(\varphi - \varphi_0) P_{n-1}^m(\cosh \sigma)$$

where φ_0 and ψ_0 are arbitrary constants.

(B. Riemann and C. Neumann.)

§ 67. *The equation of the conduction of heat.* The partial differential equation

$$\frac{\partial V}{\partial t} = \kappa \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) \quad (1)$$

plays a prominent part in Fourier's theory of the conduction of heat, and is usually called the equation of the conduction of heat. The quantity V represents the temperature and κ a constant. The same partial differential equation also occurs in the theory of diffusion.

When V is independent of y and z , we have the simple equation

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial x^2} \quad (2)$$

Let us form a few solutions of this equation which are polynomials in x and t . The simplest are

$$x \quad x^2 + 2\kappa t \quad x^3 + 6\kappa x t \quad x^4 + 12\kappa x^2 t + 24\kappa^2 t^2$$

Each solution has the form of a polynomial in x^2 and t multiplied by either 1 or x . Let us take $s = \frac{x^2}{4\kappa t}$ as a new variable; and express equation (2) in terms of s and t . The new equation is

$$s \frac{\partial^2 V}{\partial s^2} + \left(\frac{1}{2} + s\right) \frac{\partial V}{\partial s} - t \frac{\partial V}{\partial t} = 0 \quad (3)$$

This equation possesses solutions of the form $V = t^\nu \varphi(s)$ if φ satisfies the differential equation

$$s \frac{d^2 \varphi}{ds^2} + \left(\frac{1}{2} + s\right) \frac{d\varphi}{ds} - \nu \varphi = 0 \quad (4)$$

Putting $\varphi = e^{-s} \psi$, we get

$$s \frac{d^2 \psi}{ds^2} + \left(\frac{1}{2} - s\right) \frac{d\psi}{ds} - (\nu + \frac{1}{2}) \psi = 0$$

Comparing this with the equation

$$s \frac{d^2 F}{ds^2} + (\gamma - s) \frac{dF}{ds} - \alpha F = 0$$

satisfied by the function $F(\alpha, \gamma, s)$, we find that a possible value of φ is

$$\varphi = e^{-s} F\left(\nu + \frac{1}{2}; \frac{1}{2}; s\right)$$

Hence the function

$$V = t^\nu e^{-\frac{x^2}{4\kappa t}} F\left(\nu + \frac{1}{2}; \frac{1}{2}; \frac{x^2}{4\kappa t}\right) \quad (5)$$

satisfies the differential equation (2). In particular, when $\nu = -\frac{1}{2}$, we may take $\psi = 1$ and obtain the solution

$$V = t^{-\frac{1}{2}} e^{-\frac{x^2}{4\kappa t}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The solutions (5) and (6) are not polynomials in x and t .

To obtain a solution which is a polynomial in x^2 and t , we write $\sigma = -s$ in equation (4). The new equation

$$\sigma \frac{d^2\varphi}{d\sigma^2} + \left(\frac{1}{2} - \sigma\right) \frac{d\varphi}{d\sigma} + \nu\varphi = 0 \quad . \quad . \quad . \quad . \quad (7)$$

possesses the polynomial solution

$$\varphi = F(-\nu; \frac{1}{2}; \sigma)$$

when $\nu + 1$ is a positive integer. In this case the solution

$$V = t^\nu F\left(-\nu; \frac{1}{2}; -\frac{x^2}{4\kappa t}\right) \quad . \quad . \quad . \quad . \quad (8)$$

is a polynomial in x^2 and t . Differentiating with regard to x , we obtain a second solution

$$W = \frac{\partial V}{\partial x} = -\frac{1}{2} x t^{\nu-1} \frac{dF}{d\sigma}$$

Now $\frac{d\varphi}{d\sigma}$ satisfies the differential equation

$$\sigma \frac{d^2u}{d\sigma^2} + \left(\frac{3}{2} - \sigma\right) \frac{du}{d\sigma} + (\nu - 1)u = 0$$

one solution of which is given by $u = F(1 - \nu; \frac{3}{2}; \sigma)$; hence a second solution of equation (2) is given by

$$V = x t^{\nu-1} F\left(1 - \nu; \frac{3}{2}; -\frac{x^2}{4\kappa t}\right) \quad . \quad . \quad . \quad . \quad (9)$$

This is a polynomial in x and t when ν is a positive integer.

Comparing the solutions (5) and (8), we notice that if the expression (8) is written in the form $V = f(x, t)$, then the function

$$U = t^{-\frac{1}{2}} e^{-\frac{x^2}{4\kappa t}} f\left(\frac{x}{t}, -\frac{1}{t}\right) \quad . \quad . \quad . \quad . \quad (10)$$

is of the form (5), but the value of ν is different. This suggests the following theorem, which is due to Brill and Appell:

If $V = f(x, t)$ is one solution of equation (2), then the function U defined by (10) is also a solution.

To prove this, we notice that the transformation

$$x_1 = \frac{x}{t} \quad t_1 = -\frac{1}{t}$$

makes

$$s_1 = \frac{x_1^2}{4\kappa t_1} = -\frac{x^2}{4\kappa t} = -s$$

Hence equation (3) becomes

$$s_1 \frac{\partial^2 V}{\partial s_1^2} + \left(\frac{1}{2} - s_1\right) \frac{\partial V}{\partial s_1} - t_1 \frac{\partial V}{\partial t_1} = 0$$

Putting $V = t_1^{\frac{1}{2}} e^{\epsilon} W$, we find that W satisfies the equation

$$s_1 \frac{\partial^2 W}{\partial s_1^2} + \left(\frac{1}{2} + s_1\right) \frac{\partial W}{\partial s_1} - t_1 \frac{\partial W}{\partial t_1} = 0$$

which is of the same form as (3); hence the theorem is proved.

Let us next suppose that the function V in equation (1) is independent of z ; we then have the equation

$$\frac{\partial V}{\partial t} = \kappa \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad \dots \quad (11)$$

Let $x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad s = \frac{\rho^2}{4\kappa t}$

Equation (11) may then be written

$$\frac{\partial V}{\partial t} = \kappa \left(\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} \right)$$

and this is satisfied by $V = G(t, s, \varphi)$

if
$$s \frac{\partial^2 G}{\partial s^2} + (1 + s) \frac{\partial G}{\partial s} - t \frac{\partial G}{\partial t} + \frac{1}{4s} \frac{\partial^2 G}{\partial \varphi^2} = 0 \quad \dots \quad (12)$$

Now write $G = t^\nu \cos(m\varphi + \epsilon) u(s)$; the preceding equation is then satisfied if

$$s \frac{d^2 u}{ds^2} + (1 + s) \frac{du}{ds} - \left(\nu + \frac{m^2}{4s} \right) u = 0$$

This equation is transformed by the substitution $u = s^\nu v$ into

$$s \frac{d^2 v}{ds^2} + (s + 2\nu + 1) \frac{dv}{ds} + \left(\nu - \nu + \frac{4\nu^2 - m^2}{4s} \right) v = 0$$

Let us take $\nu = \pm \frac{1}{2}m$; then the term in $\frac{1}{s}$ disappears, and we obtain

$$s \frac{d^2 v}{ds^2} + (s \pm m + 1) \frac{dv}{ds} + (\pm \frac{1}{2}m - \nu) v = 0$$

Writing $s = -\sigma$, we obtain the equation satisfied by the function

$$V = F(\pm \frac{1}{2}m - \nu; 1 \pm m; \sigma)$$

Hence equation (11) is satisfied by

$$V = t^\nu \left(\frac{\rho^2}{4t} \right)^{\frac{m}{2}} F\left(\frac{m}{2} - \nu; 1 + m; -\frac{\rho^2}{4\kappa t}\right) \cos(m\varphi + \epsilon)$$

that is, by

$$V = t^\nu \rho^m F\left(-n; 1 + m; -\frac{\rho^2}{4\kappa t}\right) \cos(m\varphi + \epsilon) \quad (13)$$

When $1 - n$ and $1 - m$ are positive integers, this is a polynomial in x, y and t .

Let us now write $t_1 = -\frac{1}{t}$, $s_1 = -s$ as before; then equation (12) becomes

$$s_1 \frac{\partial^2 G}{\partial s_1^2} + (1 - s_1) \frac{\partial G}{\partial s_1} - t_1 \frac{\partial G}{\partial t_1} + \frac{1}{4s_1} \frac{\partial^2 G}{\partial \varphi^2} = 0$$

and this is satisfied by $G = t_1 e^{s_1 H}$ if

$$s_1 \frac{\partial^2 H}{\partial s_1^2} + (1 + s_1) \frac{\partial H}{\partial s_1} - t_1 \frac{\partial H}{\partial t_1} + \frac{1}{4s_1} \frac{\partial^2 H}{\partial \varphi^2} = 0$$

which is of the same form as (12). Hence we have the following theorem due to Brill:

If $V = \Psi(x, y, t)$ is a solution of (11), the function

$$V = \frac{1}{t} e^{-\frac{x^2+y^2}{4\kappa t}} \Psi\left(\frac{x}{t}, \frac{y}{t}, -\frac{1}{t}\right) \quad \dots \quad (14)$$

is also a solution.

In particular, we deduce from (13) that the function

$$V = \rho^m t^{-m-n-1} e^{-\frac{\rho^2}{4\kappa t}} F\left(-n; 1+m; \frac{\rho^2}{4\kappa t}\right) \cos(m\varphi + \varepsilon) \quad (15)$$

is a solution of equation (11).

Again, if Ψ is independent of t , a solution of (11) is given by

$$\Psi = f(x + iy) + g(x - iy)$$

where f and g are arbitrary functions; hence we may deduce that the function

$$V = \frac{1}{t} e^{-\frac{x^2+y^2}{4t}} \left[f\left(\frac{x+iy}{t}\right) + g\left(\frac{x-iy}{t}\right) \right] \quad \dots \quad (16)$$

satisfies equation (11).

In the general case, when V depends on x, y, z and t , we may make the substitution

$$x = r \sin \theta \cos \varphi \quad y = r \sin \theta \sin \varphi \quad z = r \cos \theta \quad s = \frac{r^2}{4\kappa t}$$

The transformed equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} = \frac{1}{s} \frac{\partial V}{\partial t}$$

is satisfied by $V = G(s, t) \Theta(\theta) \Phi(\varphi)$

$$\text{if} \quad \frac{d^2 \Phi}{d\varphi^2} + m^2 \Phi = 0$$

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \left[n(n+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0$$

$$s \frac{\partial^2 G}{\partial s^2} + \left(\frac{3}{2} + s \right) \frac{\partial G}{\partial s} - t \frac{\partial G}{\partial t} - \frac{n(n+1)}{4s} G = 0$$

Now put $G = t^\nu s^{\frac{n}{2}} u(s)$; then u satisfies the equation

$$s \frac{d^2 u}{ds^2} + (n + s + \frac{3}{2}) \frac{du}{ds} + \left(\frac{n}{2} - \nu \right) u = 0$$

and we find as before that there is a solution of the form

$$u = F\left(\frac{n}{2} - \nu, n + \frac{3}{2}, -s\right)$$

The equations for Φ and Θ are satisfied by $\cos(m\varphi + \epsilon)$ and $P_n^m(\cos \theta)$ respectively; hence we find that equation (1) possesses solutions of the form

$$V = t^\nu \left(\frac{r^2}{4xt} \right)^{\frac{n}{2}} F\left(\frac{n}{2} - \nu, n + \frac{3}{2}, -\frac{r^2}{4xt}\right) \cos(m\varphi + \epsilon) P_n^m(\cos \theta) \quad (17)$$

By a simple extension of the method used before, we can establish the following theorem due to Brill:

If $V = f(x, y, z, t)$ is a solution of equation (1), the function

$$V = t^{-\frac{3}{2}} e^{-\frac{r^2}{4kt}} f\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, -\frac{1}{t}\right) \quad (18)$$

is also a solution.

In particular, if we take f to be a constant, we obtain the solution

$$V = t^{-\frac{3}{2}} e^{-\frac{r^2}{4kt}} \quad (19)$$

while we may deduce from formula (17) that the function

$$V = r^n t^{-(n+\kappa+\frac{1}{2})} F\left(-\kappa, n + \frac{3}{2}, \frac{r^2}{4xt}\right) e^{-\frac{r^2}{4kt}} \cos(m\varphi + \epsilon) P_n^m(\cos \theta)$$

is a solution of (1), where $\kappa = \nu - \frac{n}{2}$ is an arbitrary constant.

References.

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 E. Cunningham, *Proc. Roy. Soc.* Vol. 81 (1908), p. 310.
 J. Brill, *Messenger of Mathematics* (1891).
 P. Appell, *Liouville's Journal*, Ser. 4, t. 8 (1894).

EXAMPLES.

1. Prove that if $V = f(x, t, \tau)$ is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t \partial \tau}$$

the function

$$V = t^{-\frac{1}{2}} f\left(\frac{x}{t}, \tau - \frac{x^2}{4t}, -\frac{1}{t}\right)$$

is also a solution, and that if $V = f(x, y, t, \tau)$ is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial t \partial \tau}$$

the function $V = \frac{1}{t} f\left(\frac{x}{t}, \frac{y}{t}, \tau - \frac{x^2 + y^2}{4t}, -\frac{1}{t}\right)$

is also a solution.

2. Obtain corresponding theorems for the equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

respectively.

§ 68. *Harmonic equations.** A partial differential equation is said to be *harmonic* when it possesses an infinite number of particular solutions, each of which is the product of a number of functions of one variable, the variable being different for each factor. Thus the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$$

is a harmonic equation, because it possesses an infinite number of particular solutions of the type

$$z = \sin \lambda x \cdot \sin \lambda y$$

where λ is an arbitrary parameter.

An equation of type

$$\frac{\partial^2 z}{\partial x \partial y} = [\varphi(x + y) - \psi(x - y)]z \quad . \quad . \quad . \quad (1)$$

will be regarded as the *canonical form* of a harmonic linear partial differential equation of the second order in two variables. It possesses an infinite number of *harmonic solutions* of the form

$$z = f(x + y) \cdot g(x - y) \quad . \quad . \quad . \quad (2)$$

where $f(s)$ and $g(t)$ satisfy the ordinary linear differential equations

$$\frac{d^2 f}{ds^2} = [\varphi(s) + \lambda]f$$

$$\frac{d^2 g}{dt^2} = [\psi(t) + \lambda]g$$

in which λ is an arbitrary parameter.

* The analysis in §§ 68 and 69 is taken from Darboux's *Théorie des Surfaces*, t. 2.

Many differential equations which occur in mathematical physics can be reduced to the canonical form which has just been given. The equations in question are of the type

$$P(s) \frac{\partial^2 w}{\partial s^2} + Q(s) \frac{\partial w}{\partial s} + R(s)w = L(t) \frac{\partial^2 w}{\partial t^2} + M(t) \frac{\partial w}{\partial t} + N(t)w \quad (3)$$

To reduce to the canonical form, we write $w = zu(\xi)v(\eta)$, where u, v, ξ , and η are determined by the equations

$$\begin{aligned} P \left(\frac{d\xi}{ds} \right)^2 &= 1 & 2 \frac{du}{d\xi} + Pu \frac{d^2 \xi}{ds^2} + Qu \frac{d\xi}{ds} &= 0 \\ L \left(\frac{d\eta}{dt} \right)^2 &= 1 & 2 \frac{dv}{d\eta} + Lv \frac{d^2 \eta}{dt^2} + Mv \frac{d\eta}{dt} &= 0 \end{aligned}$$

The equation then becomes

$$\frac{\partial^2 z}{\partial \xi^2} + z \left[\frac{1}{u} \frac{d^2 u}{d\xi^2} - \frac{2}{u^2} \left(\frac{du}{d\xi} \right)^2 + R \right] = \frac{\partial^2 z}{\partial \eta^2} + z \left[\frac{1}{v} \frac{d^2 v}{d\eta^2} - \frac{2}{v^2} \left(\frac{dv}{d\eta} \right)^2 + R \right]$$

The substitution $x = \xi + \eta$, $y = \xi - \eta$ now reduces the equation to the canonical form.

Example. To reduce

$$\frac{\partial^2 w}{\partial s^2} + \frac{n}{s} \frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial t^2} + \frac{m}{t} \frac{\partial w}{\partial t}$$

to the canonical form, we write $w = s^{-\frac{n}{2}} t^{-\frac{m}{2}} z$. The equation then becomes

$$\frac{\partial^2 z}{\partial s^2} - \frac{n(n-2)}{4s^2} z = \frac{\partial^2 z}{\partial y^2} - \frac{m(m-2)}{4t^2} z$$

The canonical form is thus

$$\frac{\partial^2 z}{\partial x \partial y} = z \left[\frac{n(n-2)}{4(x+y)^2} - \frac{m(m-2)}{4(x-y)^2} \right]$$

§ 69. *The condition that a given equation can be reduced to the harmonic form.* Let the given equation be

$$\frac{\partial^2 z}{\partial x \partial y} = zF(x, y) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we make the substitution

$$x' = \int \frac{dx}{\sqrt{X}} \quad y' = \int \frac{dy}{\sqrt{Y}}$$

the new equation will have the harmonic form if

$$\sqrt{XY} \cdot F(x, y) = \varphi(x' + y') - \psi(x' - y')$$

$$\text{i.e. if} \quad \frac{\partial^2}{\partial x'^2} (F\sqrt{XY}) = \frac{\partial^2}{\partial y'^2} (F\sqrt{XY})$$

Transforming to the variables x, y , we get

$$\sqrt{X} \frac{\partial}{\partial x} \left[\sqrt{X} \frac{\partial}{\partial x} (F \sqrt{XY}) \right] = \sqrt{Y} \frac{\partial}{\partial y} \left[\sqrt{Y} \frac{\partial}{\partial y} (F \sqrt{XY}) \right]$$

Expanding this, we get

$$2X \frac{\partial^2 F}{\partial x^2} + 3 \frac{dX}{dx} \frac{\partial F}{\partial x} + \frac{d^2 X}{dx^2} F = 2Y \frac{\partial^2 F}{\partial y^2} + 3 \frac{dY}{dy} \frac{\partial F}{\partial y} + \frac{d^2 Y}{dy^2} F$$

The problem is thus reduced to that of finding if a function X of x and a function Y of y exist, for which the above equation is satisfied.

If the equation is satisfied for two different systems (X_1, Y_1) and (X_2, Y_2) , of values of X and Y , it is satisfied for the values $(aX_1 + bX_2, aY_1 + bY_2)$ of X and Y , where a and b are arbitrary constants.

Hence when an equation is reducible to the harmonic form in two distinct ways, it is reducible to the harmonic form in an infinite number of ways.

It is sometimes interesting to determine F when X and Y are given; we may then regard the preceding equation as a harmonic equation for the determination of F . If $X = x(1 - x)$, $Y = y(1 - y)$, this equation is

$$\begin{aligned} 2x(1 - x) \frac{\partial^2 F}{\partial x^2} + 3(1 - 2x) \frac{\partial F}{\partial x} - 2F \\ = 2y(1 - y) \frac{\partial^2 F}{\partial y^2} + 3(1 - 2y) \frac{\partial F}{\partial y} - 2F \end{aligned}$$

A particular solution is $F = (1 - 2x)(1 - 2y)$; hence the equation

$$\frac{\partial^2 z}{\partial x \partial y} = z(1 - 2x)(1 - 2y)$$

can be reduced to the harmonic form by the substitution

$$x = \sin^2 \frac{1}{2} \xi \quad y = \sin^2 \frac{1}{2} \eta$$

In the case of the equation

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{m(1 - m)}{(x - y)^2} z \quad \dots \quad (5)$$

which is the canonical form of the equation

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 \zeta}{\partial \tau^2} + \frac{2m}{\tau} \frac{\partial \zeta}{\partial \tau} \quad z = \tau^m \zeta \quad \dots \quad (6)$$

the functions $X(x)$ and $Y(y)$ must satisfy the relation

$$12(X - Y) - 6(x - y) \left(\frac{dX}{dx} + \frac{dY}{dy} \right) + (x - y)^2 \left(\frac{d^2 X}{dx^2} - \frac{d^2 Y}{dy^2} \right) = 0$$

Differentiating twice with respect to x and twice with respect to y , we find that

$$\frac{d^4 X}{dx^4} = \frac{d^4 Y}{dy^4}$$

Hence $X = ax^4 + bx^3 + cx^2 + dx + e$

$$Y = ay^4 + by^3 + cy^2 + dy + e$$

the coefficients being the same in each case, since $X = Y$ when $x = y$.

By taking different biquadratics, we may obtain various transformations of the differential equation (5). In the first place, if we put

$$\begin{aligned} X &= 4x & Y &= 4y \\ x &= x'^2 & y &= y'^2 \end{aligned}$$

the transformed equation is

$$\frac{\partial^2 z}{\partial x' \partial y'} = \left[\frac{m(m-1)}{(x' + y')^2} - \frac{m(m-1)}{(x' - y')^2} \right] z \quad (7)$$

This result tells us that the equation (6) is transformed by the substitution

$$t = s^2 + \sigma^2 \quad \tau = 2s\sigma$$

into the form
$$\frac{\partial^2 \zeta}{\partial s^2} + \frac{2m}{s} \frac{\partial \zeta}{\partial s} = \frac{\partial^2 \zeta}{\partial \sigma^2} + \frac{2m}{\sigma} \frac{\partial \zeta}{\partial \sigma}$$

Secondly, if we put

$$X = 4x(1-x) \quad Y = 4y(1-y) \quad x = \sin^2 x' \quad y = \sin^2 y'$$

the equation becomes

$$\frac{\partial^2 z}{\partial x' \partial y'} = \left[\frac{m(m-1)}{\sin^2(x' + y')} - \frac{m(m-1)}{\sin^2(x' - y')} \right] z$$

and consequently possesses harmonic solutions of type

$$z = f(x' + y')f(x' - y')$$

where $f(u)$ satisfies the equation

$$\frac{d^2 f}{du^2} = \left[\frac{m(m-1)}{\sin^2 u} + \lambda \right] f \quad (8)$$

When $\lambda = 0$, this is transformed by the substitution $\cot u = i\mu$ into Legendre's equation

$$\frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{\partial f}{\partial \mu} \right\} + m(m-1)f = 0$$

When m is an integer, a solution of (8) can be expressed in finite terms. This is seen most easily by using the following theorem, which is due to Darboux : *

If we know how to solve the differential equation

$$\frac{d^2 y}{dt^2} = [\varphi(t) + h]y \quad (9)$$

for all values of the parameter h , and $y = v(t)$ is a solution when h has the value h_1 , the general solution of the equation

$$\frac{d^2 z}{dt^2} = \left[v \frac{d^2}{dt^2} \left(\frac{1}{v} \right) + h - h_1 \right] z \quad (10)$$

* *Comptes Rendus*, t. 94 (1882), p. 1456 ; *Théorie des Surfaces*, t. 2, p. 196.

is given by
$$z = \frac{dy}{dt} - \frac{y}{v} \frac{dv}{dt} \quad \dots \quad (11)$$

where y is the general solution of equation (9). This is easily verified by direct substitution.

Let us now apply this theorem to the equation

$$\frac{d^2y}{dt^2} = hy \quad \dots \quad (12)$$

When $h = -1$, this is satisfied by $y = \sin t$. Let us take this for our function v ; then we may conclude that the general solution of the equation

$$\frac{d^2z}{dt^2} = \left[\frac{1}{\sin^2 t} + h \right] z \quad \dots \quad (13)$$

is
$$z = A\sqrt{h}e^{\sqrt{h}t} - B\sqrt{h}e^{-\sqrt{h}t} - \cot t[Ae^{\sqrt{h}t} + Be^{-\sqrt{h}t}]$$

where A and B are arbitrary constants.

For particular values of h , equation (13) possesses the solutions $\sin^2 t$, $\sin^2 t \cos t$; consequently the theorem may be applied again. It will be convenient, however, to consider the more general equation

$$\frac{d^2y}{dt^2} = y \left[\frac{m(m-1)}{\sin^2 t} + \frac{n(n-1)}{\cos^2 t} + h \right] \quad \dots \quad (14)$$

When $h = -(m+n)^2$, this equation possesses the particular solution

$$y = \sin^m t \cos^n t = v, \text{ say}$$

Employing this solution, we obtain the new equation

$$\frac{d^2z}{dt^2} = z \left[v \frac{d^2}{dt^2} \left(\frac{1}{v} \right) + h + (m+n)^2 \right]$$

or
$$\frac{d^2z}{dt^2} = z \left[\frac{m(m+1)}{\sin^2 t} + \frac{n(n+1)}{\cos^2 t} + h \right] \quad \dots \quad (15)$$

This is of the same type as (14), but with $m+1$, $n+1$ written in place of m and n . It should be noticed that if $n = 0$, so that the term $\frac{n(n-1)}{\cos^2 t}$ is zero in (14), the term $\frac{n(n+1)}{\cos^2 t}$ is zero in (15); hence the solution of the equation

$$\frac{d^2z}{dt^2} = z \left[\frac{m(m+1)}{\sin^2 t} + h \right]$$

may be derived by successive steps when m is an integer. If we put $n = 1$ in (14), the term $\frac{n(n-1)}{\cos^2 t}$ in (14) is zero, but the term $\frac{n(n+1)}{\cos^2 t}$ in (15) is not zero. It is clear, then, that if both m and n are integers, the general solution of (14) can be obtained with the aid of Darboux's theorem.

If in equation (14) we write $y = \sin^m t \cos^n t u(t)$ and $s = \sin^2 t$, we find that u satisfies the hypergeometric equation

$$s(1-s) \frac{d^2 u}{ds^2} + [\gamma - (\alpha + \beta + 1)s] \frac{du}{ds} - \alpha\beta u = 0$$

where

$$2\alpha = m + n + \sqrt{-h} \quad 2\beta = m + n - \sqrt{-h} \quad 2\gamma = 2m + 1$$

Hence we have the result that the solution of the hypergeometric equation can be expressed in finite terms when

$$\gamma - \frac{1}{2} \quad \text{and} \quad \alpha + \beta$$

are integers.

EXAMPLES.

1. Prove that Laplace's equation possesses solutions of the form

$$V = \rho^m \cos m\varphi F_m(s) F_m(\sigma)$$

where $s^2 + \sigma^2 = iz$, $2s\sigma = \rho$ and $F_m(s)$ is a solution of the equation

$$\frac{d^2 F}{ds^2} + \frac{2m+1}{s} \frac{dF}{ds} + F = 0$$

Hence show that there are harmonic functions of type

$$V = J_m(s) J_m(\sigma) \cos m\varphi$$

where $J_m(s)$ is a solution of Bessel's equation.

2. If z is a solution of the harmonic equation

$$\frac{\partial^2 z}{\partial x \partial y} = [\varphi(x+y) - \psi(x-y)]z$$

the function

$$z' = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - 2(\varphi + \psi)z$$

is also a solution. (Darboux.)

3. Prove that the equation

$$\frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial V}{\partial \mu} \right] - \frac{m^2}{1 - \mu^2} V = \frac{\partial}{\partial \nu} \left[(1 - \nu^2) \frac{\partial V}{\partial \nu} \right] - \frac{m^2}{1 - \nu^2} V \quad (1)$$

is transformed by the substitution

$$\mu + \nu = \xi \quad \frac{1 + \mu\nu}{\mu + \nu} = \eta$$

into the equation

$$\xi^2 \frac{\partial^2 V}{\partial \xi^2} + 2\xi \frac{\partial V}{\partial \xi} + \frac{\partial}{\partial \eta} \left[(1 - \eta^2) \frac{\partial V}{\partial \eta} \right] - \frac{m^2}{1 - \eta^2} V = 0 \quad (2)$$

and that the equation

$$\begin{aligned} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial V}{\partial \mu} \right] - \left[\frac{m^2}{2(1 - \mu)} + \frac{k^2}{2(1 + \mu)} \right] V \\ = \frac{\partial}{\partial \nu} \left[(1 - \nu^2) \frac{\partial V}{\partial \nu} \right] - \left[\frac{m^2}{2(1 - \nu)} + \frac{k^2}{2(1 + \nu)} \right] V \end{aligned}$$

is transformed by the same substitution into

$$\xi^2 \frac{\partial^2 V}{\partial \xi^2} + 2\xi \frac{\partial V}{\partial \xi} + \frac{\partial}{\partial \eta} \left[(1 - \eta^2) \frac{\partial V}{\partial \eta} \right] - \left[\frac{m^2}{2(1 - \eta)} + \frac{k^2}{2(1 + \eta)} \right] V = 0$$

4. Prove that equation (1) of Ex. 3 is also transformed into equation (2) by the substitution

$$\mu^2 + \nu^2 - 1 = \xi^2 \qquad \mu\nu = \xi\eta$$

Hence show that equation (1) is invariant when the variables μ, ν are transformed to μ', ν' , where

$$\begin{aligned} \mu'\nu' - \mu\nu + 1 &= 0 \\ \mu'^2 + \nu'^2 + \mu'\nu' &= \mu^2 + \nu^2 - \mu\nu \end{aligned}$$

and that equation (2) is invariant under the substitution

$$\begin{aligned} \xi\eta &= \xi'\eta' - 1 \\ \xi^2 + \xi\eta &= \xi'^2 - \xi'\eta' \end{aligned}$$

INTEGRATION IN SERIES

$$\varphi(x, y, p) = 0$$
$$\frac{dy}{dx} = \mathbf{F}(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$F(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{m,n} (x - x_0)^m (y - y_0)^n \quad . \quad . \quad (2)$$
$$y - y_0 = \sum_{n=1}^{\infty} c_n (x - x_0)^n \quad . \quad . \quad . \quad (3)$$
$$\left. \begin{aligned} c_1 &= A_{00} \\ 2c_2 &= A_{10} + A_{01}c_1 \\ 3c_3 &= A_{20} + A_{11}c_1 + A_{02}c_1^2 + A_{01}c_2 \\ &\vdots \\ nc_n &= \Psi[A_{n0}, A_{n-1, 1}, \dots, c_1, c_2, \dots, c_{n-1}] \end{aligned} \right\} \dots (4)$$

Where Ψ is some polynomial in the arguments indicated, the exact form of Ψ will not be required.

It will be noticed that these equations enable us to calculate $c_1, c_2, c_3 \dots$ successively and in just one way; hence there is only one series of type (3) which gives a formal solution of the differential equation (1). We must now show that the series (3) converges for sufficiently small values of $|x - x_0|$ when the coefficients are determined by equations (4). For this purpose we require the following lemma.

If, whenever $|x - x_0| \leq a, |y - y_0| \leq b$, we have the inequality

$$|F(x, y)| \leq M$$

where M is a constant, and (where both real and complex values of x and y are considered) we also have the inequality

$$|A_{m,n}| \leq \frac{M}{a^m b^n} \dots \dots \dots (5)$$

To prove this, we write $x = x_0 + ae^{i\theta}, y = y_0 + be^{i\phi}$ and multiply (2) by $e^{-im\theta - in\phi}$.

Then, since the power series (2) is uniformly convergent for $|x - x_0| \leq a, |y - y_0| \leq b$, we may integrate it term by term with regard to θ and ϕ between the limits 0 and 2π , and obtain

$$\int_0^{2\pi} \int_0^{2\pi} F(x_0 + ae^{i\theta}, y_0 + be^{i\phi}) e^{-im\theta - in\phi} d\theta d\phi = 4\pi^2 A_{m,n} a^m b^n \quad (6)$$

the other terms disappearing on account of the relation

$$\int_0^{2\pi} \int_0^{2\pi} e^{ip\theta + iq\phi} d\theta d\phi = 0 \quad \left. \begin{array}{l} p \neq 0 \\ q \neq 0 \end{array} \right\}$$

Since the modulus of each element of the double integral on the left-hand side of (6) is less than the corresponding element of the integral

$$\int_0^{2\pi} \int_0^{2\pi} M d\theta d\phi = 4\pi^2 M^2$$

the truth of the inequality (5) is manifest.

We now consider the auxiliary differential equation

$$\frac{dz}{dx} = G(x, z) \dots \dots \dots (7)$$

where

$$G(x, z) = \left. \begin{array}{l} \frac{M}{\left(1 - \frac{x - x_0}{a}\right) \left(1 - \frac{z - z_0}{b}\right)} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{M}{a^m b^n} (x - x_0)^m (z - z_0)^n \end{array} \right\} \dots \dots (8)$$

When we endeavour to satisfy this differential equation by a power series of type

$$z - z_0 = \sum_{n=1}^{\infty} g_n (x - x_0)^n \dots \dots \dots (9)$$

we are led to a system of equations of exactly the same form as (4), except that c_n is replaced throughout by g_n and A_n by the *positive quantity* $\frac{M}{a^m b^n}$. It is clear that this system of equations will determine a *positive* value for g_n , because all the coefficients in the polynomials Ψ are positive. If now we compare the two systems of equations and take into account the inequality (5), it is easily seen that

$$|c_n| \leq g_n$$

for all values of n . Consequently, if we can prove that the power series (9) converges for sufficiently small values of $|x - x_0|$, the series (3) will converge for such values of $|x - x_0|$.

Now the auxiliary differential equation (7) is easily solved, for it may be written in the form

$$\left(1 - \frac{z - z_0}{b}\right) dz = \frac{M dx}{1 - \frac{x - x_0}{a}}$$

consequently, the solution is

$$b \left(1 - \frac{z - z_0}{b}\right)^2 = 2aM \log \left(1 - \frac{x - x_0}{a}\right) + C$$

where C is an arbitrary constant. The condition $z = z_0$, when $x = x_0$, is satisfied by taking $C = b$, and then we have

$$z - z_0 = b \left[1 - \sqrt{1 + \frac{2aM}{b} \log \left(1 - \frac{x - x_0}{a}\right)}\right]$$

Now the function on the right-hand side can be developed in a power series of type (9) for sufficiently small values of $|x - x_0|$. This is seen most easily by reverting the equation

$$x - x_0 = a \left[1 - \exp \left\{-\frac{z - z_0}{aM} + \frac{(z - z_0)^2}{2abM}\right\}\right] \quad (10)$$

and using a known theorem on the reversion of series.

Let us assume, then, that the power series which is obtained in this way converges for $|x - x_0| \leq \rho$. Since it is a solution of the differential equation (7), its coefficients can be calculated in the manner described, and are consequently all positive. If σ is the smaller of the two quantities a and ρ , the series (3) will converge for $|x - x_0| < \sigma$, and it can consequently be differentiated term by term. The existence of a solution y of the differential equation (1) is thus established. This solution satisfies the supplementary conditions:

1°. $y = y_0$ when $x = x_0$.

2°. y is an analytic function of x in the neighbourhood of $x = x_0$.

It is clearly the only solution of the differential equation which satisfies these conditions.

The existence theorem does not give us a satisfactory estimate of the magnitude of the radius of convergence of the series (3), for it frequently happens that the radius of convergence of this series is much greater than that of the series (9). To illustrate this point, let us consider the differential equation

$$\frac{dy}{dx} = \frac{y+x}{y-x} \equiv F(x, y)$$

The general solution is $y = x \pm \sqrt{c^2 + 2x^2}$; consequently, the solution which satisfies the condition that $y = 1$ when $x = 0$ is

$$y = x + \sqrt{1 + 2x^2}$$

and in this case the radius of convergence of the series (3) is $\frac{1}{\sqrt{2}}$.

Now, if we write $F(x, y)$ in the form $\frac{1 + (y-1) + x}{1 + (y-1) - x}$, we see that $F(x, y)$ can certainly be expanded in a convergent series of powers of x , and $y-1$ if $|x| < \frac{1}{3}$ and $|y-1| < \frac{1}{3}$. A suitable value of M is in this case not less than 3, for if we put $x = \frac{1}{3}$, $y = \frac{2}{3}$, we find that $F = 3$.

To determine the radius of convergence of the series (9), we find when $\frac{dz}{dx}$ is infinite, i.e. when $\frac{dx}{dz}$ is zero. Differentiating (10), we find that $\frac{dx}{dz} = 0$ when $z - z_0 = b$, that is, when

$$x - x_0 = a \left(1 - e^{-\frac{b}{2aM}} \right)$$

Consequently, ρ is certainly not greater than $\left| a \left(1 - e^{-\frac{b}{2aM}} \right) \right|$. Substituting the particular values $a = b = \frac{1}{3}$, $M = 3$, we get

$$\rho \leq \frac{1}{3} (1 - e^{-\frac{1}{2}}) \leq \frac{1}{18}.$$

This is considerably less than $\frac{1}{\sqrt{2}}$, the radius of convergence of the series (3).

Cauchy's existence theorem enables us to determine the value of y for values of x which differ from x_0 by quantities whose moduli are less than some number σ . To obtain the value of y for other values of x , we must adopt a process of continuation. If x_p is the value of x for which the value of y is required, the method is to proceed from x_0 to x_p by a series of short steps. Starting at x_0 with the assigned value y_0 , we can calculate the value y_1 of y for $x = x_1$ by the above method, provided $|x_1 - x_0| < \sigma$.

If now $F(x, y)$ can also be expanded in a convergent power series of type

$$F(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{m,n} (x - x_1)^m (y - y_1)^n$$

we use the value of y_1 to calculate the value y_2 of y for $x = x_2$, provided $|x_2 - x_1| < \sigma_1$, where σ_1 is some positive number. In a similar way we can proceed from x_2 to x_3 , from x_3 to x_4 , and so on. If the quantities $x_1, x_2, x_3, \dots$ are suitably chosen, it may be possible in this way to calculate the value of y for $x = x_p$.

In practice, this method is very laborious, especially as the successive increments of x must generally be small in order that there may be no doubt about the convergence. In actual calculations it is customary therefore to use certain methods of approximation which will now be described.

§ 71. *Runge's method of approximate solution.* In the method of approximation invented by Runge* the value of y for $x = x_p$ is calculated by successive steps in which intermediate values $x_1, x_2, \dots, x_{p-1}$ of x are used just as in the process of continuation, but instead of calculating $y_1, y_2, \dots$ with the aid of power series, certain expressions are used which are similar in form to those used in Simpson's method of approximate integration. The approximate expression for y_1 is, in fact,

$$y_1^* = y_0 + \frac{1}{6}(\beta + 4\beta' + \beta'') \quad . \quad . \quad . \quad (11)$$

where

$$\begin{aligned} \beta &= h F(x_0, y_0) & \beta' &= h F(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}\beta) & x_1 &= x_0 + h \\ \beta'' &= h F(x_0 + h, \eta) & \eta &= y_0 + h F(x_0 + h, y_0 + \beta). \end{aligned}$$

When the series

$$y_1 - y_0 = c_1 h + c_2 h^2 + c_3 h^3 + \dots \quad . \quad . \quad . \quad (12)$$

converges, the corresponding expansion for $y_1^* - y_0$ can be shown to agree with it in the first three terms. To prove this we must show that the first three derivatives of y_1^* are equal to those of y_1 when $h = 0$.

Now it follows at once from equation (2) that

$$\begin{aligned} F(x_0, y_0) &= A_{00} & \frac{\partial F}{\partial x_0} &= A_{10} & \frac{\partial F}{\partial y_0} &= A_{01} \\ \frac{\partial^2 F}{\partial x_0^2} &= 2A_{20} & \frac{\partial^2 F}{\partial x_0 \partial y_0} &= A_{11} & \frac{\partial^2 F}{\partial y_0^2} &= A_{02} \end{aligned}$$

where F stands for $F(x_0, y_0)$; hence, when $h = 0$,

$$\begin{aligned} \left(\frac{d\beta}{dh}\right)_0 &= F(x_0, y_0) = A_{00} & \left(\frac{d^2\beta}{dh^2}\right)_0 &= 0 & \left(\frac{d^3\beta}{dh^3}\right)_0 &= 0 \\ \left(\frac{d\beta'}{dh}\right)_0 &= A_{00} \\ \left(\frac{d^2\beta'}{dh^2}\right)_0 &= \frac{\partial F}{\partial x_0} + \frac{d\beta}{dh} \frac{\partial F}{\partial y_0} = A_{10} + A_{01}A_{00} = A_{10} + c_1 A_{01} = 2c_2 \end{aligned}$$

* *Math. Ann.* Bd. 46. This method and certain graphical methods of approximation are described in C. Runge's *Graphical Methods*, New York (1912), p. 120.

$$\left(\frac{d^3\beta'}{dh^3}\right)_0 = 3 \left[\frac{1}{4} \frac{\partial^2 F}{\partial x_0^2} + \frac{1}{2} \frac{d\beta}{dh} \frac{\partial^2 F}{\partial x_0 \partial y_0} + \frac{1}{4} \left(\frac{d\beta}{dh}\right)^2 \frac{\partial^2 F}{\partial y_0^2} \right]$$

$$= \frac{3}{2} [A_{20} + c_1 A_{11} + c_1^2 A_{02}]$$

$$\left(\frac{d\beta''}{dh}\right)_0 = A_{00}$$

$$\left(\frac{d^3\beta''}{dh^3}\right)_0 = 2 \left[\frac{\partial F}{\partial x_0} + \frac{\partial F}{\partial y_0} \frac{d\eta}{dh} \right] = 2(A_{10} + A_{01}c_1) = 4c_2$$

$$\left(\frac{d^3\beta'''}{dh^3}\right)_0 = 3 \left[\frac{\partial^2 F}{\partial x_0^2} + 2 \frac{\partial^2 F}{\partial x_0 \partial y_0} \frac{d\eta}{dh} + \frac{\partial^2 F}{\partial y_0^2} \left(\frac{d\eta}{dh}\right)^2 + \frac{\partial F}{\partial y_0} \frac{d^2\eta}{dh^2} \right]$$

$$= 6[A_{20} + c_1 A_{11} + c_1^2 A_{02} + 2c_2 A_{01}]$$

Consequently, if we make use of the relations (4), we find that

$$\left(\frac{dy_1^*}{dh}\right)_0 = A_{00} = c_1 \quad \left(\frac{d^2y_1^*}{dh^2}\right)_0 = 2c_2 \quad \left(\frac{d^3y_1^*}{dh^3}\right)_0 = 6c_3$$

and this means that the first three derivatives of y_1^* agree with those of y_1 when $h = 0$. It follows from this result that the quantity y_1^* will often give a good approximation for y_1 whether the series (12) converges or not. In fact, if $y_1 = f(h)$, we may write

$$y_1 - y_0 = c_1 h + c_2 h^2 + c_3 h^3 + \frac{h^4}{4!} f''''(\theta h)$$

where θ is some positive quantity less than 1, the last term representing the remainder in Maclaurin's expansion of $f(h)$.

Other formulae of approximation have been given by Heun * and Kutta.† The latter has compared the results obtained by a number of methods of approximation by considering the case of the equation

$$\frac{dy}{dx} = \frac{y - x}{y + x} = f(x, y)$$

when the supplementary condition states that $y = 1$ when $x = 0$. The accurate solution, as found by the method of § 21, is

$$\log(x^2 + y^2) - 2 \tan^{-1} \frac{x}{y} = 0$$

The different methods to be compared may be described as follows :

$$1^\circ. f = \frac{y - x}{y + x} \quad f' = -\frac{2(x^2 + y^2)}{(y + x)^3} \quad f'' = \frac{4(x^2 + y^2)(2y - x)}{(y + x)^5}$$

$$f''' = -\frac{20(x^2 + y^2)(3y^2 - 2xy + x^2)}{(y + x)^7}$$

$$\Delta y = f\Delta x + f' \frac{(\Delta x)^2}{2!} + f'' \frac{(\Delta x)^3}{3!} + f''' \frac{(\Delta x)^4}{4!} \quad (\text{Taylor.})$$

* *Zeitschrift für Math. u. Phys.* Bd. 45.

† *Ibid.* Bd. 46.

$$2^{\circ}. \quad \Delta' = \frac{1}{2}f(x, y)\Delta x \quad \Delta'' = \frac{1}{2}f(x + \frac{1}{2}\Delta x, y + \Delta')\Delta x$$

$$\Delta''' = \frac{1}{2}f(x + \frac{1}{2}\Delta x, y + \Delta' + \Delta'')\Delta x$$

$$\Delta'''' = \frac{1}{2}f(x + \frac{3}{2}\Delta x, y + \Delta' + \Delta'' + \Delta''')\Delta x$$

$$\Delta y = \Delta' + \Delta'' + \Delta''' + \Delta'''' \quad (\text{Euler.})$$

$$3^{\circ}. \quad \Delta' = f(x, y)\Delta x \quad \Delta'' = f(x + \frac{1}{3}\Delta x, y + \frac{1}{3}\Delta')\Delta x$$

$$\Delta''' = f(x + \frac{2}{3}\Delta x, y + \frac{2}{3}\Delta'')\Delta x$$

$$\Delta y = \frac{\Delta' + 3\Delta'''}{4} \quad (\text{Heun.})$$

$$4^{\circ}. \quad \Delta' = f(x, y)\Delta x \quad \Delta'' = f(x + \frac{1}{2}\Delta x, y + \frac{1}{2}\Delta')\Delta x$$

$$\Delta''' = f(x + \Delta x, y + \Delta')\Delta x \quad \Delta'''' = f(x + \Delta x, y + \Delta''')\Delta x$$

$$\Delta y = \frac{\Delta' + 4\Delta'' + \Delta''''}{6} \quad (\text{Runge.})$$

$$5^{\circ}. \quad \Delta' = f(x, y)\Delta x \quad \Delta'' = f\left(x + \frac{\Delta x}{3}, y + \frac{\Delta'}{3}\right)\Delta x$$

$$\Delta''' = f(x + \frac{2}{3}\Delta x, y + \Delta'' - \frac{1}{3}\Delta')\Delta x$$

$$\Delta'''' = f(x + \Delta x, y + \Delta''' - \Delta'' + \Delta')\Delta x$$

$$\Delta y = \frac{\Delta' + 3\Delta'' + 3\Delta''' + \Delta''''}{8} \quad (\text{Kutta.})$$

Let the value of y be required for $x = 1$. The interval $(0, 1)$ will be divided up into three smaller intervals in which $\Delta x = 0.2, 0.3$, and 0.5 respectively. The value of Δy must then be calculated for each of these intervals by the various methods of approximation. If $\Delta_1 y, \Delta_2 y, \Delta_3 y$ are the calculated values of the three increments, the approximate value of y for $x = 1$ is

$$y_1 = 1 + \Delta_1 y + \Delta_2 y + \Delta_3 y$$

The values of $\Delta_1 y, \Delta_2 y, \Delta_3 y$, as calculated by the different methods, are shown in the following table. The last column contains the values of Δy calculated by taking $\Delta x = 1$:

	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$	$\Delta_0 y$
Taylor	·1666667	·3368533	·4936913	
Euler	·1754353	·3573505	·5367900	·62242
Heun	·1680250	·3395806	·4990390	·51613
Runge	·1678487	·3393690	·4991167	·52381
Kutta	·1678449	·3392158	·4982940	·49914
True value	·1678417	·3392094	·4982784	·49828

Some applications of the method of approximation are given in R. Emden's *Gaskugeln*, Leipzig (1906). W. J. Harrison (*Cambr. Phil. Trans.*, Vol. XXII (1913), p. 39) has recently used Runge's method to solve the equation

$$\frac{dp}{d\theta} = \frac{A}{z^2} \left(1 - \frac{B}{zp} \right)$$

where $z = 1 + c \cos \theta$ and A , B and C are constants. This equation occurs in the theory of lubrication. Lord Kelvin has given a method of approximating to a solution of the equation

$$x^4 \frac{d^2 u}{dx^2} + u^* = 0$$

(*Phil. Mag.*, March, 1887, p. 291 ; July, 1908, p. 1.)

For other references see *Encyklopädie der Mathematischen Wissenschaften*, Band II 3, Heft 2 (1915), p. 150. Some approximate methods of solving linear partial differential equations are given by Lord Rayleigh, *Phil. Mag.*, Vol. 22, 1911, p. 225 ; W. Ritz, *Crelle's Journal*, Bd. 135 (1909), p. 1 ; L. F. Richardson, *Phil. Trans. A.*, Vol. 210 (1911), p. 307.

EXAMPLES.

1. A solution of the equation

$$\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$$

is to satisfy the condition that $y = 1$ when $x = 0$; find its value when $x = \frac{1}{2}$ by dividing the range $(0, \frac{1}{2})$ into two intervals for which $\Delta x = 0.2$ and 0.3 respectively and using Runge's method of approximation. (A. R. Forsyth.)

2. The sides of a triangle ABC vary according to the equation

$$\frac{dy}{dx} = \frac{x(y - x)}{y - xy}$$

where
$$x = \frac{c + a - b}{a + b - c} \quad y = \frac{b + c - a}{a + b - c}$$

If we have initially $a = 3$, $b = 2$, $c = 2$, so that $x = 1$, $y = \frac{1}{2}$, calculate the values of y for $x = 1.2, 1.8, 2.4, 3.0, 3.6, 4.4, 5.2$ respectively, and show that in the last case we have approximately

$$\frac{a}{13} = \frac{b}{4} = \frac{c}{13}$$

so that the triangle is again isosceles. (F. E. Hackett.)

§ 72. *The representation of solutions of linear differential equations by means of power series.* This method is particularly successful when a given differential equation can be written in the form

$$[x^m F(\mathfrak{s}) + G(\mathfrak{s})]y = 0 \quad . \quad . \quad . \quad (1)$$

where $\mathfrak{s}$ denotes the operator $x \frac{d}{dx}$ and m is a positive integer. It is easy to see that an equation of the form

$$F(\mathfrak{s})y + H(\mathfrak{s}) \frac{d^m y}{dx^m} = 0 \quad . \quad . \quad . \quad (2)$$

can be expressed in the form (1), for we have the equations

$$\left. \begin{aligned} x^m H(s)u &= H(s-m)x^m u \\ x^m \frac{d^m y}{dx^m} &= s(s-1) \dots (s-m+1)y \end{aligned} \right\} \dots (3)$$

provided $H(s)$ is a polynomial in s . The second relation has already been proved. To verify the first we put $x = e^t$, $s = \frac{d}{dt} = D$; we have then to show that

$$e^{mt} H(D)u = H(D-m)e^{mt} \cdot u$$

Let $u = e^{-mt}v$, $H(D) = \sum b_s D^s$; then, by Leibnitz's theorem,

$$\begin{aligned} D^s e^{-mt}v &= e^{-mt} \left[\frac{d^s v}{dt^s} - s \cdot m \frac{d^{s-1}v}{dt^{s-1}} + \frac{s(s-1)}{1 \cdot 2} m^2 \frac{d^{s-2}v}{dt^{s-2}} \dots \right] \\ &= e^{-mt} (D-m)^s v \end{aligned}$$

Hence the theorem follows.

If now we multiply equation (2) by x^m , we see from (3) that it may be written in the form (1), where

$$G(s) = H(s-m)s(s-1) \dots (s-m+1)$$

Let us now endeavour to satisfy equation (1) by a power series of form

$$y = \sum_{n=0}^{\infty} a_n x^{p+mn} \dots (3)$$

Substituting in the differential equation and making use of the relation $s x^p = p x^p$, we find that

$$\left. \begin{aligned} a_0 G(p) &= 0 \\ a_0 F(p) + a_1 G(p+m) &= 0 \\ a_1 F(p+m) + a_2 G(p+2m) &= 0 \\ \dots \dots \dots \end{aligned} \right\} \dots (4)$$

The equation $G(p) = 0$ is called the *indicial equation*; it is clear that if this equation is satisfied, the coefficient a_0 may be chosen arbitrarily. Equations (4) then determine $a_1, a_2, \dots$ uniquely, provided none of the quantities $G(p+sm)$ vanish

$$\text{Since } \frac{a_{n+1}}{a_n} = - \frac{F(p+n-1m)}{G(p+nm)}$$

it appears that the power series converges for all values of x when the degree of the polynomial G is greater than that of F . For, if the degree of G is $s+x$ and that of F is s , we find that when n is very large $\frac{a_{n+1}}{a_n}$ is equal to n^{-x} multiplied by a factor which is approximately constant. When the degree of F is greater than that of G , the power series is always divergent, and the method fails. When

F and G are of the same degree, the series generally converge for any value of x whose modulus is less than some fixed number.

When the indicial equation $G(\rho) = 0$ has p distinct roots, no two of which differ by an integer, the method provides us with p distinct solutions of the differential equation, assuming, of course, that the series converge. When two or more roots are equal or differ by integers, the number of solutions which can be expressed simply as power series may be less than the degree of $G(\rho)$. In such a case it is possible to obtain other solutions involving logarithmic terms as well as powers of x .

§ 73. *The hypergeometric equation.* The hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\frac{dy}{dx} - \alpha\beta y = 0 \quad (1)$$

can be written in the form

$$(\mathfrak{s} + \gamma)\frac{dy}{dx} - (\mathfrak{s} + \alpha)(\mathfrak{s} + \beta)y = 0$$

and also in the form

$$x(\mathfrak{s} + \alpha)(\mathfrak{s} + \beta)y - \mathfrak{s}(\mathfrak{s} + \gamma - 1)y = 0$$

The indicial equation is now $\rho(\rho + \gamma - 1) = 0$, and the equations determining the coefficients are

$$\begin{aligned} \rho(\rho + \gamma - 1)a_0 &= 0 \\ (\rho + \alpha)(\rho + \beta)a_0 &= (\rho + 1)(\rho + \gamma)a_1 \\ (\rho + \alpha + 1)(\rho + \beta + 1)a_1 &= (\rho + 2)(\rho + \gamma + 1)a_2 \\ &\dots \end{aligned}$$

Hence, if $F(\alpha, \beta; \gamma; x)$ be used to denote the hypergeometric series

$$1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1}x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)1 \cdot 2}x^2 + \dots$$

we generally have the two independent solutions,

$$\begin{aligned} y_1 &= F(\alpha, \beta; \gamma; x) \\ y_2 &= x^{1-\gamma}F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; x) \end{aligned}$$

It is clear that the first series terminates when either α or β is zero or a negative integer, provided γ is not a numerically smaller negative integer. The second series terminates when either $\gamma - \alpha$ or $\gamma - \beta$ is a positive integer, provided $\gamma - 1$ is not a numerically smaller positive integer.

When γ is an integer and one of the series does not terminate, the method fails to furnish us with two independent solutions. Let us first of all deal with the case in which γ is a positive integer and neither α nor β is a negative integer. It is convenient to make use

of Euler's Gamma function $\Gamma(z)$, which possesses the characteristic properties

$$\begin{aligned}\Gamma(1+z) &= z\Gamma(z) & \Gamma(z)\Gamma(1-z) &= \pi \operatorname{cosec} \pi z \\ \Gamma(n) &= (n-1)! & \Gamma(1-n) &= \infty\end{aligned}$$

where n is a positive integer. In virtue of the last equation, it appears that the two series

$$\begin{aligned}& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{p=0}^{\infty} \frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{\Gamma(\gamma+p)\Gamma(1+p)} x^p \\ & \frac{1}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1-\gamma+r)\Gamma(\beta+1-\gamma+r)}{\Gamma(2-\gamma+r)\Gamma(1+r)} x^{1-\gamma+r}\end{aligned}$$

become identical when γ is a positive integer, for in the last series some of the terms vanish on account of the infinite factor $\Gamma(2-\gamma+r)$ in the denominator, $1-\gamma+r$ being negative for these terms. Now the series in question are simply constant multiples of $F(\alpha, \beta; \gamma; x)$ and $x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$; consequently we have two solutions of the hypergeometric equation which are usually independent, but which become identical when γ is a positive integer. Let us now put $\gamma = n + \varepsilon$ where n is a positive integer, and consider the limiting value of the expression

$$\begin{aligned}& \frac{1}{\varepsilon} \left[\sum_{p=0}^{\infty} \frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{\Gamma(\gamma+p)\Gamma(1+p)} x^p \right. \\ & \quad \left. - \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1-\gamma+r)\Gamma(\beta+1-\gamma+r)}{\Gamma(2-\gamma+r)\Gamma(1+r)} x^{1-\gamma+r} \right]\end{aligned}$$

when $\varepsilon \rightarrow 0$. Since the expression satisfies the hypergeometric equation when ε is not zero, we can expect its limiting value for $\varepsilon = 0$ to be also a solution of the hypergeometric equation.

Now, since

$$\Gamma(1-\varepsilon) = (-\varepsilon)(-\varepsilon-1)\dots(-\varepsilon-n+p+1)\Gamma(-\varepsilon-n+p+1)$$

we have $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \Gamma(-\varepsilon-n+p+1)} = (-1)^{n-p}(n-p-1)! \quad n > p$

Also

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{\Gamma(n+p+\varepsilon)\Gamma(1+p)} x^p - \frac{\Gamma(\alpha+p-\varepsilon)\Gamma(\beta+p-\varepsilon)}{\Gamma(1+p-\varepsilon)\Gamma(n+p)} x^{p-\varepsilon} \right]$$

is easily seen to be equal to

$$\begin{aligned}& \frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{\Gamma(n+p)\Gamma(1+p)} \\ & \times \left[\log x + \frac{\Gamma'(\alpha+p)}{\Gamma(\alpha+p)} + \frac{\Gamma'(\beta+p)}{\Gamma(\beta+p)} - \frac{\Gamma'(n+p)}{\Gamma(n+p)} - \frac{\Gamma'(1+p)}{\Gamma(1+p)} \right] x^p\end{aligned}$$

where $\Gamma'(z)$ denotes the derivative of $\Gamma(z)$.

Again, it follows from the relation

$$\Gamma(\alpha + p) = (\alpha + p - 1) \dots \alpha \Gamma(\alpha)$$

that
$$\frac{\Gamma'(\alpha + p)}{\Gamma(\alpha + p)} = \frac{1}{\alpha + p - 1} + \frac{1}{\alpha + p - 2} + \dots \frac{1}{\alpha} + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

hence, we finally obtain as the limiting value of our expression,

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(n)} F(\alpha, \beta; n; x) \left[\log x + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\beta)}{\Gamma(\beta)} - \frac{\Gamma'(n)}{\Gamma(n)} - \frac{\Gamma'(1)}{\Gamma(1)} \right] + F_1(\alpha, \beta; n; x)$$

where

$$\begin{aligned} F_1(\alpha, \beta; n; x) = & \sum_{p=1}^{\infty} x^p \frac{\Gamma(\alpha + p)\Gamma(\beta + p)}{\Gamma(n + p)\Gamma(1 + p)} \\ & \times \left[\frac{1}{\alpha} + \dots \frac{1}{\alpha + p - 1} + \frac{1}{\beta} + \dots \right. \\ & \left. \frac{1}{\beta + p - 1} - \frac{1}{1} - \dots \frac{1}{p} - \frac{1}{n} - \dots \frac{1}{n + p - 1} \right] \\ & - \sum_{r=0}^{n-2} (-1)^{n-r-1} \frac{\Gamma(\alpha + 1 - n - r)\Gamma(\beta + 1 - n - r)}{\Gamma(1 + r)} x^{1-n+r} \end{aligned}$$

When γ and $1 - \alpha$ are positive integers, we may obtain two independent solutions of the hypergeometric equation in the following way. We have already seen that the equation possesses the particular solution

$$y_1 = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} \frac{d^{-\alpha}}{dx^{-\alpha}} [x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma}] = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma, x)$$

and can be derived from the differential equation

$$\left. \begin{aligned} x(1-x) \frac{d^2 u}{dx^2} + [\alpha - \gamma + 2 + (\beta - \alpha - 3)x] \frac{du}{dx} \\ + (\beta - \alpha - 1)u = 0 \end{aligned} \right\} \quad (2)$$

which is satisfied by $u = x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma}$ by a suitable process of transformation. If, then, we find a second solution of this last equation, a second solution of the hypergeometric equation can be derived from it by the transformation.

Putting $u = x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma}v$, and using the method of § 44, we find that a second solution of the differential equation (2) is

$$u = x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma} \int \frac{dx}{x^{\gamma-\alpha}(1-x)^{\beta-\gamma+1}}$$

hence a second solution of the hypergeometric equation is

$$y_2 = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} \frac{d^{-\alpha}}{dx^{-\alpha}} \left[x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma} \int \frac{dx}{x^{\gamma-\alpha}(1-x)^{\beta-\gamma+1}} \right]$$

Expanding the integrand in ascending powers of x and integrating, we see that this solution is of the form

$$y_2 = A \log x \cdot F(\alpha, \beta; \gamma; x) + \sum_{r=1}^{\infty} b_r x^r$$

where A and b_r are constants.

The case in which $\gamma - 1$ is a negative integer may be discussed by making the substitution $y = x^{1-\gamma}w$, when it is found that w satisfies a hypergeometric equation in which α, β, γ are replaced by $\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma$ respectively. Since $2 - \gamma$ is now a positive integer, the preceding analysis is applicable.

Other expressions for the solutions of the hypergeometric equation may be obtained by making a change in the independent variable. If we write $1 - x$ in place of x , we find that the differential equation is also satisfied by

$$y_3 = F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - x)$$

$$y_4 = (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - x)$$

while if we make the substitution $x = \frac{1}{x'}$, we find that the equation is satisfied by the functions

$$y_5 = x^{-\alpha} F\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; \frac{1}{x}\right)$$

$$y_6 = x^{-\beta} F\left(\beta, \beta - \gamma + 1; \beta - \alpha + 1; \frac{1}{x}\right)$$

By combining these transformations, we can also obtain series of powers of $1 - \frac{1}{x}$, $\frac{1}{1 - x}$ and $\frac{x}{x - 1}$ respectively. The problem of finding the relations between the different solutions which are obtained in this way is of some interest. For a discussion of this problem the reader is referred to Forsyth's *Differential Equations* and to a memoir by Dr. E. W. Barnes, *Proc. London Math. Soc.* Ser. 2, Vol. 6 (1908), p. 141.

EXAMPLES.

1. Prove that

$$F(\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma; x)$$

$$F(\alpha, \beta; \gamma; x) = (1 - x)^{-\beta} F\left(\beta, \gamma - \alpha, \gamma, \frac{x}{x - 1}\right)$$

2. Show that the hypergeometric equation can be reduced to the normal form

$$\frac{d^2 v}{dx^2} + \frac{v}{4} \left[\frac{1 - \lambda^2}{x^2} + \frac{1 - \mu^2}{(1 - x)^2} + \frac{1 + \mu^2 - \lambda^2 - v^2}{x(1 - x)} \right] = 0$$

where $\lambda^2 = (1 - \gamma)^2$ $\mu^2 = (\alpha - \beta)^2$ $v^2 = (\gamma - \alpha - \beta)^2$

3. Verify that $F(-n, \beta; \beta; -x) = (1+x)^n$

$$2F\left(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \frac{1}{2}; x^2\right) = (1+x)^n + (1-x)^n$$

$$xF(1, 1; 2; -x) = \log(1+x)$$

$$F\left(\frac{1}{2}n, -\frac{1}{2}n; \frac{1}{2}; \sin^2 x\right) = \cos nx$$

$$F\left(\frac{n+1}{2}, \frac{1-n}{2}; \frac{3}{2}; \sin^2 x\right) = \frac{\sin nx}{nx}$$

$$F\left(1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; \sin^2 x\right) = \frac{\sin nx}{n \sin x \cos x}$$

$$F\left(\frac{n+1}{2}, \frac{1-n}{2}; \frac{1}{2}; \sin^2 x\right) = \frac{\cos nx}{\cos x}$$

$$F\left(\frac{n+1}{2}, \frac{n}{2}; n+1; x\right) = 2^n \{1 + \sqrt{1-x}\}^{-n}$$

4. Obtain the recurrence formulae

$$F(\alpha, \beta+1; \gamma; x) - F(\alpha, \beta; \gamma; x) = \frac{\alpha x}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x)$$

$$F(\alpha+1, \beta+1; \gamma; x) - F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta x}{\gamma^2} F(\alpha+1, \beta+1; \gamma+1; x)$$

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x)$$

§ 74. *A limiting case of the hypergeometric equation.* If we put $x = \frac{z}{\beta}$ in the hypergeometric equation and make $\beta \rightarrow \infty$, we obtain the equation

$$z \frac{d^2 y}{dz^2} - (z - \gamma) \frac{dy}{dz} - \alpha y = 0$$

which may be written in the forms

$$(\mathfrak{s} + \gamma) \frac{dy}{dz} - (\mathfrak{s} + \alpha)y = 0$$

$$\mathfrak{s}(\mathfrak{s} + \gamma - 1)y - z(\mathfrak{s} + \alpha)y = 0$$

where $\mathfrak{s} = z \frac{d}{dz}$. This equation possesses the solutions

$$y_1 = F(\alpha; \gamma; z)$$

$$y_2 = z^{1-\gamma} F(\alpha - \gamma + 1; 2 - \gamma; z)$$

where $F(\alpha; \gamma; z)$ denotes Kummer's function

$$1 + \frac{\alpha}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)1 \cdot 2} z^2 + \dots$$

the series being convergent for all values of z . When γ is an integer, the two solutions are not independent, and a second solution can be derived from the second solution of the hypergeometric equation.

EXAMPLES.

1. Prove that $F(\alpha; \gamma; x) = e^x F(\gamma - \alpha; \gamma; x)$ (Kummer.)
2. Prove that if no two of the quantities $\alpha_1 \dots \alpha_m, \gamma_1 \dots \gamma_n, 1$ are equal or differ by an integer, the equation

$$\{(\mathfrak{S} + \alpha_1)(\mathfrak{S} + \alpha_2) + \dots (\mathfrak{S} + \alpha_m) - \frac{1}{x} \mathfrak{S}(\mathfrak{S} + \gamma_1 - 1) \dots (\mathfrak{S} + \gamma_n - 1)\} y = 0$$
 is satisfied by the series

$$\begin{aligned}
 y &= F(\alpha_1, \alpha_2, \dots \alpha_m; \gamma_1, \gamma_2, \dots \gamma_n; x) \\
 &= 1 + \frac{\alpha_1 \cdot \alpha_2 \dots \alpha_m}{1 \cdot \gamma_1 \cdot \gamma_2 \dots \gamma_n} x + \frac{\alpha_1(\alpha_1 + 1)\alpha_2(\alpha_2 + 1) \dots \alpha_m(\alpha_m + 1)}{1 \cdot 2 \cdot \gamma_1(\gamma_1 + 1)\gamma_2(\gamma_2 + 1) \dots \gamma_n(\gamma_n + 1)} x^2 + \dots
 \end{aligned}$$

and by n other series of type,

$$\begin{aligned}
 y_1 &= x^{1-\gamma_1} F(\alpha_1 - \gamma_1 + 1, \alpha_2 - \gamma_1 + 1, \dots \alpha_m - \gamma_1 + 1; \\
 &\quad 2 - \gamma_1, \gamma_2 - \gamma_1 + 1, \dots \gamma_n - \gamma_1 + 1; x)
 \end{aligned}$$

If $m > n$, the series converge for all values of x , but if $m = n + 1$ the series converge, for $|x| < 1$. If $m > n + 1$, the series diverge. (Pochhammer.)

§ 75. *Bessel's equation.* Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0 \quad . \quad . \quad . \quad (1)$$

may be written in the form

$$(\mathfrak{S}^2 - \nu^2)y + x^2 y = 0$$

The indicial equation is now $\rho^2 - \nu^2 = 0$, and so there are two solutions

$$\begin{aligned}
 y_1 &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+\nu}}{2^{2r+\nu} r! \Gamma(r + \nu + 1)} = J_{\nu}(x) \\
 y_2 &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r-\nu}}{2^{2r-\nu} r! \Gamma(r - \nu + 1)} = J_{-\nu}(x)
 \end{aligned}$$

These series converge for all finite values of x .

When n is an integer, we have $J_{-n}(x) = (-1)^n J_n(x)$, and so the two solutions already obtained are not independent. To obtain a second solution in this case, we choose a number $\nu = n + \varepsilon$ very nearly equal to n and consider the limiting value of

$$\frac{1}{\varepsilon} [J_{\nu}(x) - (-1)^n J_{-\nu}(x)]$$

as $\varepsilon \rightarrow 0$. This limit, which is usually denoted by the symbol $\pi Y_n(x)$ can be written in the form

$$\pi Y_n(x) = \left[\frac{\partial}{\partial \nu} J_{\nu}(x) \right]_{\nu=n} - (-1)^n \left[\frac{\partial}{\partial \nu} J_{-\nu}(x) \right]_{\nu=n}$$

Making use of the relations

$$\begin{aligned}\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \Gamma(-\epsilon - n + p + 1)} &= (-1)^{n-p} (n - p - 1)! \\ \frac{d}{d\nu} \left(\frac{x}{2}\right)^{2r+\nu} &= \left(\frac{x}{2}\right)^{2r+\nu} \log \left(\frac{x}{2}\right) \\ \frac{d}{d\nu} \left(\frac{x}{2}\right)^{2r+2n-\nu} &= -\left(\frac{x}{2}\right)^{2r+2n-\nu} \log \left(\frac{x}{2}\right) \\ \lim_{\nu \rightarrow n} \frac{\partial}{\partial \nu} \left[\frac{(-1)^r}{r! \Gamma(r + \nu + 1)} - (-1)^n \frac{(-1)^{r+n}}{(r+n)! \Gamma(r+n-\nu+1)} \right] \\ &= \frac{(-1)^{r+1}}{r! (r+n)!} \left[\frac{\Gamma'(r+n+1)}{\Gamma(r+n+1)} + \frac{\Gamma'(r+1)}{\Gamma(r+1)} \right] \\ \frac{\Gamma'(r+1)}{\Gamma(r+1)} &= 1 + \frac{1}{2} + \dots + \frac{1}{r} + \frac{\Gamma'(1)}{\Gamma(1)} = \Psi(r) - \gamma\end{aligned}$$

where $\gamma = -\frac{\Gamma'(1)}{\Gamma(1)}$, we finally obtain

$$\begin{aligned}\pi Y_n(x) &= 2 \left[\log \left(\frac{x}{2}\right) + \gamma \right] J_n(x) - \sum_{p=0}^{n-1} \frac{\Gamma(n-p)}{\Gamma(p+1)} \left(\frac{x}{2}\right)^{2p-n} \\ &\quad - \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r)! r!} [\Psi(r+n) + \Psi(r)] \left(\frac{x}{2}\right)^{n+2r}\end{aligned}$$

When ν is not an integer, it is convenient to define the function $Y_\nu(x)$ by the equation

$$Y_\nu(x) = \operatorname{cosec} \nu\pi [\cos \nu\pi \cdot J_\nu(x) - J_{-\nu}(x)]$$

then, as $\nu \rightarrow n$, $Y_\nu(x) \rightarrow Y_n(x)$. In all cases $J_\nu(x)$ and $Y_\nu(x)$ may be taken as two independent solutions of equation (1).

The function $K_\nu(x)$, which has been used previously, can be defined by the equation

$$K_\nu(ix) = -\frac{i\pi}{2} e^{-\frac{i\nu\pi}{2}} [J_\nu(x) - iY_\nu(x)]$$

The function $e^{-\frac{i\nu\pi}{2}} J_\nu(ix)$ is usually denoted by the symbol $I_\nu(x)$; it satisfies the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0$$

If we write $w = e^{-x} I_\nu(x)$, we find that w satisfies the equation

$$x^2 \frac{d^2 w}{dx^2} + x \frac{dw}{dx} - (x^2 + \nu^2) e^x w = 0$$

or
$$x^2 \frac{d^2 w}{dx^2} + (2x^2 + x) \frac{dw}{dx} + (x - \nu^2) w = 0$$

Writing this equation in the form

$$(\nu^2 - \nu^2)w + x(2\nu + 1)w = 0$$

and applying the method of § 72, we see that there are two power series satisfying the equation. One starts with x^ν and the other with $x^{-\nu}$; it is clear, then, that if ν is positive, the former must be a constant multiple of $e^{-x}I_\nu(x)$. Denoting the power series by

$\sum_{n=0}^{\infty} a_n x^{\nu+n}$, we have the recurrence formula

$$[(\nu + n)^2 - \nu^2]a_n + (2\nu + 2n - 1)a_{n-1} = 0$$

Hence it follows that $e^{-x}I_\nu(x)$

$$= \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \left[1 - \frac{2\nu + 1}{1(1 + 2\nu)}x + \frac{(2\nu + 1)(2\nu + 3)}{1 \cdot 2 \cdot (2\nu + 1)(2\nu + 2)}x^2 - \dots \right]$$

There is an important relation between $J_\nu(x)$ and $Y_\nu(x)$, which is easily proved. Let us write $y = J_\nu(x)$, $z = Y_\nu(x)$; then, if we multiply the equations

$$\begin{aligned} \frac{d}{dx} \left[x \frac{dy}{dx} \right] + \left(x - \frac{\nu^2}{x} \right) y &= 0 \\ \frac{d}{dx} \left[x \frac{dz}{dx} \right] + \left(x - \frac{\nu^2}{x} \right) z &= 0 \end{aligned}$$

by z and y respectively and subtract, we get

$$\begin{aligned} \frac{d}{dx} \left[x \left(z \frac{dy}{dx} - y \frac{dz}{dx} \right) \right] &= 0 \\ \therefore z \frac{dy}{dx} - y \frac{dz}{dx} &= \frac{A}{x} \dots \dots \dots (2) \end{aligned}$$

where A is some constant. To determine the value of the constant, we make use of the formula

$$z = Y_\nu(x) = \operatorname{cosec} \nu\pi [\cos \nu\pi J_\nu(x) - J_{-\nu}(x)]$$

and the expressions for $J_\nu(x)$, $J_{-\nu}(x)$. The coefficient of $\frac{1}{x}$ on the left-hand side of (2) is consequently

$$- \frac{\operatorname{cosec} \nu\pi}{\Gamma(1 + \nu)\Gamma(1 - \nu)} [\nu + \nu] = A$$

Now

$$\Gamma(1 + \nu) = \nu\Gamma(\nu)$$

and

$$\Gamma(\nu)\Gamma(1 - \nu) = \frac{\pi}{\sin \nu\pi}$$

therefore

$$A = - \frac{2}{\pi}$$

consequently

$$y \frac{dz}{dx} - x \frac{dz}{dy} = \frac{2}{\pi x}$$

This relation still holds when ν is an integer.

EXAMPLES.

1. Prove that the functions $J_\nu(x)$, $Y_\nu(x)$ satisfy the recurrence relations

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$\frac{dJ_\nu}{dx} = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x)$$

$$\frac{dJ_\nu}{dx} = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x)$$

$$\frac{dJ_\nu}{dx} = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)]$$

$$\frac{d^2 J_\nu}{dx^2} = \frac{1}{4}[J_{\nu+2}(x) + J_{\nu-2}(x) - 2J_\nu(x)]$$

2. Prove that the function $F_n(x) = e^{-x} I_n(x)$ satisfies the equation

$$\frac{dF_n}{dx} = \frac{1}{2}[F_{n-1} + F_{n+1} - 2F_n]$$

3. Prove that the equation

$$\frac{d^2 v}{dx^2} + a^2 x^m v = 0$$

is satisfied by $v = x^{\frac{1}{2}}[AJ_n(ax) + BY_n(ax)]$

where $z = 2nx - \frac{1}{2n}$.

4. Prove that

$$J_\nu(x) \cos x = \frac{(2x)^\nu}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu + 2n + \frac{1}{2})}{(2n)! \Gamma(2\nu + 2n + 1)} (2x)^{2n}$$

$$J_\nu(x) \sin x = \frac{(2x)^\nu}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu + 2n + \frac{3}{2})}{(2n+1)! \Gamma(2\nu + 2n + 2)} (2x)^{2n+1}$$

5. Prove that $Y_{\nu-1}(x)J_\nu(x) - Y_\nu(x)J_{\nu-1}(x) = \frac{2}{\pi x}$

§ 76. *Legendre's equation.* If we put $x = \frac{1}{z}$ in Legendre's equation

$$(1 - x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + n(n+1)u = 0$$

it takes the form

$$z^2(z^2 - 1) \frac{d^2 u}{dz^2} + 2z^3 \frac{du}{dz} + n(n+1)u = 0$$

or

$$z^2 \mathfrak{z}(\mathfrak{z} + 1)u - (\mathfrak{z} + n)(\mathfrak{z} - n - 1)u = 0$$

The indicial equation is now $(\rho + n)(\rho - n - 1) = 0$, and we have the two solutions

$$u_1 = z^{-n} - \frac{n(n-1)}{2(2n-1)}z^{-n+2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)}z^{-n+4} - \dots$$

$$u_2 = z^{n+1} + \frac{(n+1)(n+2)}{2(2n+3)}z^{n+3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4(2n+3)(2n+5)}z^{n+5} + \dots$$

which may be written in the form

$$u_1 = x^n F\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2} - n; x^{-2}\right)$$

$$u_2 = x^{-n-1} F\left(\frac{n+1}{2}, \frac{n+2}{2}; n + \frac{3}{2}; x^{-2}\right)$$

with the usual notation of the hypergeometric function. When n is a positive integer, the first series terminates, and we obtain two independent solutions, which are usually chosen as follows:

$$u_1 = P_n(x) = \frac{2^n x^n \Gamma(n + \frac{1}{2})}{\Gamma(n+1) \sqrt{\pi}} F\left(\frac{1-n}{2}; -\frac{n}{2}; \frac{1}{2} - n; x^{-2}\right)$$

$$u_2 = Q_n(x) = \frac{\sqrt{\pi}}{2^{n+1}} \frac{\Gamma(n+1)}{\Gamma(n + \frac{3}{2})} x^{-n-1} F\left(\frac{n+1}{2}, \frac{n+2}{2}; n + \frac{3}{2}; x^{-2}\right)$$

The last series converges when $|x| > 1$.

When $n + \frac{1}{2}$ is an integer, there is only one power series in $\frac{1}{x}$ which satisfies the equation; a second solution for this case may be derived from the second solution of the hypergeometric equation when γ is an integer.

It should be noticed that Legendre's equation is transformed into a hypergeometric equation by putting $1 - x = 2s$; hence the function $F\left(n+1, -n; 1; \frac{1-x}{2}\right)$ satisfies Legendre's equation.

EXAMPLES.

1. Prove that Legendre's equation is satisfied by the coefficient of a^n in the expansion of $(1 - 2ax + a^2)^{-\frac{1}{2}}$ in ascending powers of a . Hence show that this coefficient is $P_n(x)$.

2. Prove that $P_n(x)$ and $Q_n(x)$ satisfy the recurrence formulae of type

$$nP_n - 2(n-1)xP_{n-1} + (n-1)P_{n-2} = 0$$

$$P'_n - xP'_{n-1} = nP_{n-1}$$

$$xP'_n - P'_{n-1} = nP_n$$

where primes denote differentiations with respect to x .

§ 77. *Two convergence theorems.* The method of integration by means of power series can be applied to differential equations of a much more general character than those which have already been considered, but the recurrence formulae for determining the coefficients are rather complicated. There are, however, two general convergence theorems which will be given here for the case of a linear differential equation of the second order.

If $P = p_0 + p_1x + p_2x^2 + \dots$, $Q = q_0 + q_1x + q_2x^2 + \dots$ (1) are two power series which converge for $|x| < R$, the differential equation

$$\frac{d^2y}{dx^2} = P \frac{dy}{dx} + Qy \dots \dots \dots (2)$$

has solutions of the type $y = A_0 + A_1x + A_2x^2 + \dots$, where A_0 and A_1 are arbitrary, and the other coefficients are determined uniquely in terms of them by the recurrence formula

$$n(n-1)A_n = (n-1)A_{n-1}p_0 + (n-2)A_{n-2}p_1 + \dots A_1p_{n-2} \left. \begin{array}{l} + A_{n-2}q_0 + \dots A_1q_{n-3} + A_0q_{n-2} \end{array} \right\} \quad (3)$$

It follows that if the values of y and $\frac{dy}{dx}$ are given for $x = 0$, there is one and only one solution y which can be represented by a power series of the above type.

To establish the convergence of the series for y , we remark that since the series for P and Q converge for $|x| < R$, we can find a positive number M such that

$$|p_n| < \frac{M}{r^n} \qquad |q_n| < \frac{M}{r^{n+1}}$$

where r is any pre-assigned positive number less than R . This is a particular case of the theorem proved in § 70 for the case of a power series in two variables.

These inequalities imply that

$$|(n-s)p_{s-1} + q_{s-2}| < (n-s+1) \frac{M}{r^{s-1}}$$

We now compare the recurrence formula (3) with a second one,

$$n(n-1)B_n = nB_{n-1}M + (n-1)B_{n-2} \frac{M}{r} + \dots 2B_1 \frac{M}{r^{n-2}} + B_0 \frac{M}{r^{n-1}}$$

which is such that the coefficient of B_{n-s} is positive and numerically greater than the modulus of the coefficient of A_{n-s} , and possesses the useful property that the limiting value of the ratio $\frac{B_{n-1}}{B_n}$ is easily estimated. We have, in fact,

$$n(n-1)B_n - \frac{1}{r}(n-1)(n-2)B_{n-1} = nB_{n-1}M$$

and so

$$\lim_{n \rightarrow \infty} \frac{B_{n-1}}{B_n} = r$$

It follows from this result that the series $\sum B_n x^n$ converges if $|x| < r$.

Now, if $B_0 = |A_0|$, $B_1 = |A_1|$, it is easily seen from the two recurrence formulae that $|A_n| \leq B_n$; hence it follows that the series $\sum A_n x^n$ converges if $|x| < r$. Taking $2r = |x| + R$, we find that the series converges if $|x| < R$.

Next, consider the differential equation *

$$x^2 \frac{d^2 y}{dx^2} = xP \frac{dy}{dx} + Qy \quad (4)$$

If we assume $y = x^p(A_0 + A_1 x + A_2 x^2 + \dots)$ and substitute in the differential equation, we obtain

$$[\rho(\rho - 1) - \rho p_0 - q_0]A_0 = 0$$

and a number of other equations. This first equation is the *indicial equation*. We shall suppose that it gives two distinct values of ρ . Let ρ be one of these and ρ' the other; then the recurrence formula for determining the other coefficients may be written in the form

$$n(n + \rho - \rho')A_n = \{(n - 1 + \rho)p_1 + q_1\}A_{n-1} + \dots + \{(\rho p_n + q_n)A_0\} \quad (5)$$

We now choose a positive number M , which is such that

$$|p_n| < \frac{M}{r^n} \quad |q_n| < \frac{M}{r^n}$$

where r , as before, is a pre-assigned positive number less than R .

The recurrence formula (5) may then be compared with

$$n(n - \delta)B_n = M \left\{ (n + \tau) \frac{B_{n-1}}{r} + (n - 1 + \tau) \frac{B_{n-2}}{r^2} + \dots (1 + \tau) \frac{B_0}{r^n} \right\}$$

where $\tau = |\rho|$, $\delta = |\rho - \rho'|$.

As before, the coefficient of $B_{n-\delta}$ is positive and numerically greater than that of $A_{n-\delta}$. If, then, we regard the above equation as holding only for $n > \delta$ and define $B_p = |A_p|$ for $p \leq \delta$, we shall have

$$|A_n| < B_n$$

Now

$$r\{n(n - \delta)B_n\} - (n - 1)(n - 1 - \delta)B_{n-1} = M(n + \tau)B_{n-1}$$

and so

$$\lim_{n \rightarrow \infty} \frac{B_{n-1}}{B_n} = r$$

Consequently the series $\sum B_n x^n$ and $\sum A_n x^n$ both converge for $|x| < r$, and by choosing $r = \frac{1}{2}[|x| + R]$, where $|x| < R$, we may conclude that the series $\sum A_n x^n$ converges for $|x| < R$.

The method requires modification if δ is an integer, when dealing with the smaller root ρ of the quadratic, because then the factor

* For this equation, the reader should consult the works of Fuchs, *Crelle's Journal*, Bd. 66, p. 122 (1866), Bd. 68 (1868), p. 354, and Frobenius, *Crelle's Journal*, Bd. 76 (1873), p. 214; Bd. 80 (1875), p. 317.

$n + \rho - \rho'$ becomes zero for some value of n . The complete solution of the equation may in this case be obtained by assuming

$$y = x^\rho \log x (A_0 + A_1 x + \dots) + x^\sigma (C_0 + C_1 x + \dots)$$

For an account of this method, the reader is referred to Forsyth's *Differential Equations*, 4th edn. p. 243.

It should be noticed that in the case of an equation of type (4), the point $x = 0$ is a singularity or branch point of at least one of the two linearly independent solutions of the differential equation. In other words, we cannot represent both solutions by series of type $\sum_{n=0}^{\infty} b_n x^n$, in which only positive integral powers of x occur.

It should be noticed, also, that the problem of finding a solution of the differential equation which satisfies the supplementary conditions $u = A$, $\frac{du}{dx} = B$, when $x = 0$, does not generally possess a solution. Sometimes it may possess a solution when A and B are connected by a certain relation. This occurs, for instance, when two linearly independent solutions are

$$y_1 = \sum_{n=0}^{\infty} a_n x^n \quad y_2 = x^\rho \sum_{n=0}^{\infty} b_n x^n$$

where ρ is not an integer. In order that a solution satisfying the conditions may exist, it is necessary that we should have

$$\frac{A}{a_0} = \frac{B}{a_1}$$

If $\rho < 1$, a solution which satisfies the supplementary conditions is unique, for the derivative of y_2 is then infinite for $x = 0$.

If $\rho > 1$, there are ∞' solutions which satisfy the supplementary conditions, for then both y_2 and $\frac{dy_2}{dx}$ vanish for $x = 0$; consequently, any constant multiple of y_2 may be added to a solution which is known to satisfy the supplementary conditions, and a new solution is obtained which satisfies the requirements.

The case in which a logarithmic term appears in y_2 can be discussed in a similar way.

EXAMPLES.

1. Prove that an equation which possesses the two linearly independent solutions $y = u_1(x)$, $y = u_2(x)$ is given by

$$\begin{vmatrix} \frac{d^2 y}{dx^2} & \frac{dy}{dx} & y \\ \frac{d^2 u_1}{dx^2} & \frac{du_1}{dx} & u_1 \\ \frac{d^2 u_2}{dx^2} & \frac{du_2}{dx} & u_2 \end{vmatrix} = 0$$

2. Prove that a linear differential equation of the second order, which possesses two linearly independent solutions of the form

$$y = [\exp(a_1 x^{-1} + \dots a_n x^{-n})] \left[\sum_{m=0}^{\infty} b_m x^m \right]$$

is necessarily of the form

$$x^{2n+2} P_1(x) \frac{d^2 y}{dx^2} + x^{n+1} P_2(x) \frac{dy}{dx} + P_3(x) y = 0$$

where $P_1(x)$, $P_2(x)$, $P_3(x)$ denote power series of type $\sum_{m=0}^{\infty} c_m x^m$.

3. Obtain solutions of the following equations in the form of power series :

$$\frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0$$

$$\frac{d^3 y}{dx^3} + \frac{3}{x} \frac{d^2 y}{dx^2} + \left(4 - \frac{4n^2 - 1}{x^2} \right) \frac{dy}{dx} + \frac{4}{x} y = 0$$

§ 78. *The method of successive approximations.** Let us consider the two differential equations

$$\frac{dy}{dx} = f_1(x, y, z) \quad \frac{dz}{dx} = f_2(x, y, z) \quad \dots \quad (1)$$

where $f_1(x, y, z)$, $f_2(x, y, z)$ are functions which can be represented by power series

$$f_1(x, y, z) = \sum_{m, n, p} A_{m, n, p} (x - a)^m (y - b)^n (z - c)^p$$

$$f_2(x, y, z) = \sum_{m, n, p} B_{m, n, p} (x - a)^m (y - b)^n (z - c)^p$$

which converge absolutely and uniformly in the region

$$|x - a| \leq r \quad |y - b| \leq R \quad |z - c| \leq R$$

the series may, of course, converge in a more extensive region.

We shall assume that for all points (x, y, z) of the region specified, two positive numbers M and k can be found, such that

$$|f_i(x, y, z)| \leq M \quad (i = 1, 2)$$

$$\text{and} \quad \left| \frac{\partial f_i(x, y, z)}{\partial y} \right| \leq k \quad \left| \frac{\partial f_i(x, y, z)}{\partial z} \right| \leq k \quad (i = 1, 2)$$

If, then, (x, y, z) and (x, y', z') are two arbitrary points of the region, we have

$$|f_i(x, y', z') - f_i(x, y, z)| \leq k \{ |y' - y| + |z' - z| \}$$

For we have

$$f_i(x, y', z') - f_i(x, y, z) = \int_y^{y'} \frac{\partial f_i(x, y, z')}{\partial y} dy + \int_z^{z'} \frac{\partial f_i(x, y, z)}{\partial z} dz$$

and the absolute values of the integrands never exceed k ; hence the theorem follows.

* Cf. E. Picard, *Traité d'Analyse*, t. 2, 2nd ed. (1905), p. 340. For a system of linear differential equations the method has been developed in a very convenient form by H. F. Baker: *Proc. London Math. Soc.* Ser. 1, Vol. 35 (1902), p. 337; Ser. 2, Vol. 2 (1904), p. 293; *Phil. Trans. A.*, Vol. 216 (1916), p. 129.

We shall now endeavour to determine y and z as functions of x , so that $y = b$ and $z = c$ when $x = a$.

We commence with the approximate values $y_0 = b$, $z_0 = c$, then determine functions y_1, z_1 with the aid of the equations

$$\frac{dy_1}{dx} = f_1(x, y_0, z_0) \quad \frac{dz_1}{dx} = f_2(x, y_0, z_0)$$

and the initial conditions $y_1 = b$, $z_1 = c$ when $x = a$. We next determine functions y_2, z_2 with the aid of the equations

$$\frac{dy_2}{dx} = f_1(x, y_1, z_1) \quad \frac{dz_2}{dx} = f_2(x, y_1, z_1)$$

and the initial conditions $y_2 = b$, $z_2 = c$. Continuing in this way, we obtain a series of successive approximations (y_0, z_0) , (y_1, z_1) , (y_2, z_2) , ... (y_n, z_n) We must now show that as $n \rightarrow \infty$, the functions y_n, z_n tend to limiting functions y, z , which satisfy the differential equations.

Let ρ be the smaller of the two positive quantities $r, \frac{R}{M}$, and let us limit the variable x to the region $|x - a| \leq \rho$.

We have
$$y_1 - b = \int_a^x f_1(\xi, b, c) d\xi$$

where the path of integration is the straight line joining the points a and x in the complex plane. The length of this line is $|x - a|$, and since $|f_1(\xi, b, c)| \leq M$, we have

$$|y_1 - b| \leq M|x - a| \leq M\rho \leq R.$$

Similarly we find that $|z_1 - c| \leq R$, and since $|f_1(x, y_1, z_1)|$ is also not greater than M , we find in a similar way that $|y_2 - b| \leq R$.

Hence it follows that

$$|y_n - b| \leq R \quad |z_n - c| \leq R$$

We now write

$$u_n = y_n - y_{n-1} \quad v_n = z_n - z_{n-1} \quad (n = 1, 2, 3, \dots)$$

so that

$$y_n = b + u_1 + u_2 + \dots + u_n$$

$$z_n = c + v_1 + v_2 + \dots + v_n$$

If y_n tends to a limit when $n \rightarrow \infty$, this limit is the sum of the infinite series

$$b + u_1 + u_2 + \dots$$

We shall show that this series converges absolutely and uniformly for $|x - a| \leq \rho$.

We have

$$u_n = \int_a^x [f_1(x, y_{n-1}, z_{n-1}) - f_1(x, y_{n-2}, z_{n-2})] dx$$

Now

$$|f_1(x, y_{n-1}, z_{n-1}) - f_1(x, y_{n-2}, z_{n-2})| \leq k\{|y_{n-1} - y_{n-2}| + |z_{n-1} - z_{n-2}|\}$$

hence $|f_1(x, y_1, z_1) - f_1(x, y_0, z_0)| \leq k(|u_1| + |v_1|) \leq 2kM|x - a|$

$$\text{and so } |u_2| \leq 2kM \int_0^{|x-a|} |x-a|d|x-a| = \frac{2kM|x-a|^2}{2!}$$

$$\text{Similarly, } |v_2| \leq \frac{2kM|x-a|^2}{2!}$$

Again,

$$|f_1(x, y_2, z_2) - f_1(x, y_1, z_1)| \leq k(|u_2| + |v_2|) \leq \frac{2k^2M|x-a|^2}{2!}$$

$$\text{and since } u_3 = \int_a^x [f_1(x, y_2, z_2) - f_1(x, y_1, z_1)] dx$$

$$\text{we have } |u_3| \leq \frac{(2k)^2M}{2!} \int_0^{|x-a|} |x-a|^2 d|x-a| = \frac{(2k)^2M|x-a|^3}{3!}$$

Continuing in this way, we find that

$$|u_n| \leq \frac{(2k)^{n-1}M|x-a|^n}{n!} \quad (n = 1, 2, 3, \dots)$$

Since the series with the general term $\frac{(2k)^{n-1}M\rho^n}{n!}$ converges, it follows that the series

$$b + u_1 + u_2 + u_3 + \dots$$

converges absolutely and uniformly for $|x-a| \leq \rho$. Hence y_n tends to a limit y and similarly z_n tends to a limit z , and these functions y and z are continuous functions of x .

It now follows from the equation

$$y_n = b + \int_a^x f_1(x, y_{n-1}, z_{n-1}) dx$$

$$\text{that } y = b + \int_a^x f_1(x, y, z) dx$$

and, similarly, we can prove that

$$z = c + \int_a^x f_2(x, y, z) dx$$

Hence y and z satisfy the differential equations and the specified boundary conditions.

Some particular cases of the foregoing theorem are of special importance.

1°. If $f_1(x, y, z)$ is independent of z , we have a method of solving the equation

$$\frac{dy}{dx} = f_1(x, y)$$

2°. If $f_1(x, y, z), f_2(x, y, z)$ are linear functions of y and z , we obtain two linear equations

$$\frac{dy}{dx} = A_1(x)y + B_1(x)z + C_1(x)$$

$$\frac{dz}{dx} = A_2(x)y + B_2(x)z + C_2(x)$$

3°. In particular, if $f_1(x, y, z) = z$, we obtain the equation

$$\frac{d^2y}{dx^2} = f_2\left(x, y, \frac{dy}{dx}\right)$$

which becomes a linear equation of the second order when $f_2(x, y, z)$ is a linear function of y and z .

The above method of approximation is closely related to a graphical method of solution which has been developed by J. Massau* and others.† If the differential equation is $p = f(x, y)$, the first step is to draw the isoclinals or curves along which f has a constant value, and to associate with each curve a line $y = fx$ drawn in a direction making an angle whose tangent is f with the axis of x . This correspondence between straight lines and isoclinals is independent of the system of rectangular coordinates, and gives a convenient graphical representation of the differential equation.

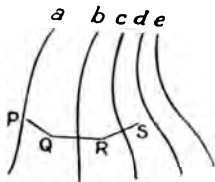


FIG. 7.

Now let a system of isoclinals $a, b, c, d, e, \dots$ be drawn at small intervals, and let the first of these pass through the point P , whose coordinates (x_0, y_0) represent the initial state of affairs.

Through P we draw a line in the direction associated with a to a point Q somewhere between a and b . Through Q we draw a line in the direction associated with b to a point R somewhere between b and c , and so on. The curve touching this broken line at its points of intersection with $a, b, c, \dots$ respectively is a first approximation.‡ To find a better approximation we introduce a rectangular system of coordinates x, y , laying the axis of x somewhat in the mean direction of the broken line. Let y_1 be the function of x corresponding to the first approximation. The second approximation y_2 is then obtained as an integral curve of $f(x, y_1)$; in other words,

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

To find y_2 we choose the new axis of y at unit distance from the old origin through which the rays were drawn, and associate the ordinates of the points in which the rays meet this axis with the abscissa of the points in which the broken line $PQR\dots$ meets the corresponding curves $a, b, c, \dots$. Plotting the points with these ordinates and abscissae as coordinates, we obtain a series of points on a curve

$$Y = f(x, y_1)$$

* 'Mémoire sur l'intégration graphique et ses applications,' *Ann. de l'Assoc. des ing. sortis des écoles spéciales de Gand* (1878), (1884), (1886), (1887), (1890).

† See M. d'Ocagne, *Calcul graphique et nomographie* (1908), p. 149; C. Runge, *Graphical Methods*, New York (1912), p. 120.

‡ This curve need not be drawn.

This curve is to be integrated graphically, beginning at the point associated with P. Adopting the method of approximate integration in which the function f is assumed to have a constant value in a series of intervals each of which contains one of x 's for points that have been plotted, we obtain a second broken line, which is treated in the same way as the first, and so on. We thus obtain a series of functions $y_1, y_2, \dots, y_n, \dots$, and according to the theory of the method of successive approximations they would converge to the true solution if the integration were exact instead of only approximate.

For a fuller account of the method and an estimate of the possible error, the reader is referred to Runge's book.

There is another method of approximation which is particularly useful * when we wish to establish the existence of a solution of the equation

$$\frac{dy}{dx} = f(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the case when the variables are real and $f(x, y)$ is not restricted as much as in § 70. The following presentation of the method is taken from the lectures given by G. A. Bliss at the Princeton Colloquium † and from Picard's *Traité d'Analyse*, t. 2, p. 322.

Let us suppose that the function $f(x, y)$ is continuous in the interior of a certain region R of the xy -plane, and such that the quotient

$$\frac{f(x, y') - f(x, y)}{y' - y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is finite when (x, y) and (x, y') lie in any closed region whose points are all interior to R . This restriction was introduced by Lipschitz.

A so-called Cauchy polygon for the equation (1) through a point (ξ, η) interior to R is defined by means of equations of the form

$$\begin{aligned} y_1 &= \eta + f(\xi, \eta)(x_1 - \xi) \\ y_2 &= y_1 + f(x_1, y_1)(x_2 - x_1) \\ y &= y_{n-1} + f(x_{n-1}, y_{n-1})(x - x_{n-1}) \end{aligned}$$

The division points $\xi < x_1 < x_2 < \dots$

may be taken for convenience at equal intervals δ from each other. Any value $x > \xi$ will lie on one of the intervals $x_{n-1}x_n$, and the polygon will either be well-defined for all such values, or else there will be a constant β such that for every x in the interval $\xi \leq x < \beta$ the points of the polygon are interior to R , while for $x = \beta$ the corresponding point (x, y) will be a point of the boundary of R . The polygon ‡ defined by the equations above may be denoted by

* See E. Picard, *Comptes Rendus*, Vol. 128 (1899), p. 1363; P. Painlevé, *Bull. de la Soc. Math. de France*, Vol. 27 (1899), p. 151.

† *The Princeton Colloquium Lectures on Mathematics*, New York (1913).

‡ i.e. the function $y = P_1(x)$.

$P_1(x)$, and the analogous one when the division points are distant $\delta/2^{n-1}$ from each other by $P_n(x)$.

A common interval $\xi \leq x \leq a$ for two functions $P(x)$, $Q(x)$ with respect to any region R may be defined as one over which both are interior to R , and one such that on any ordinate of the interval all the points between $\{x, P(x)\}$ and $\{x, Q(x)\}$ are also interior points of R .

Consider now a closed region R_1 interior to R and containing the point (ξ, η) , and let m and k be two constants greater respectively than the absolute values of $f(x, y)$ and the quotient (2) in the region R_1 . If $l > 0$ is given in advance, the partitions for any two polygons $P(x)$, $Q(x)$ through (ξ, η) can be taken so small that

$$|P(x) - Q(x)| \leq \frac{l}{k} \{e^{k|x-\xi|} - 1\}$$

for all values of x in any common interval of $P(x)$ and $Q(x)$ with respect to R_1 . For at any point (x, y) , where $y = P(x)$, the equation

$$P' = f(x, P) + \{f(x_{n-1}, y_{n-1}) - f(x, P)\} = f(x, P) + \rho$$

is satisfied by the forward and backward derivatives of the polygon P . On account of the continuity of $f(x, y)$ there exists for any l a constant μ such that the inequalities

$$|x - x'| < \mu \quad |y - y'| < \mu$$

imply that

$$|f(x, y) - f(x', y')| < l/2$$

whenever the points (x, y) and (x', y') are in R_1 . If the subdivisions for $P(x)$ are taken less than both μ and μ/m in length, it follows that on the polygon $P(x)$,

$$|x - x_{n-1}| < \mu \quad |P(x) - y_{n-1}| < m|x - x_{n-1}| < \mu$$

and hence the absolute value of ρ is less than $l/2$. Similarly $Q(x)$ satisfies an equation $Q' = f(x, Q) + \sigma$

where $|\sigma| < l/2$, provided that its intervals are less in length than both μ and μ/m . The difference $P - Q$ has forward and backward derivatives which satisfy the relations

$$\begin{aligned} |P' - Q'| &\leq |f(x, P) - f(x, Q)| + |\rho| + |\sigma| \\ &< k|P - Q| + l \end{aligned}$$

The desired inequality now follows at once when we use the following lemma, remembering that P and Q have the same initial value η at $x = \xi$. If u is a single-valued function of x with well-defined forward and backward derivatives at each point of an interval $x_0 \leq x \leq x_1$, and such that

$$|u'| < k|u| + l$$

then, for any ξ and x in the interval, u also satisfies the inequality

$$|u| \leq |u(\xi)|e^{k|x-\xi|} + \frac{l}{k} \{e^{k|x-\xi|} - 1\} \quad . \quad . \quad (3)$$

Let us write $t = |x - \xi|$, and consider the function

$$v = |v_0|e^{kt} + \frac{l}{k}(e^{kt} - 1) \quad (0 \leq t \leq t_1)$$

satisfying the differential equation

$$\dot{v} = kv + l$$

and having $|v_0| = |u(\xi)|$ as its initial value when $t = 0$. The value of u is never greater than that of v , since otherwise the difference $u - v$ would vanish and have a positive or vanishing forward derivative at some point. At a point where u and v are equal, however,

$$|\dot{u}| < k|u| + l = kv + l = \dot{v}$$

which is a contradiction. This proves the lemma.

If now $P(x)$ is a polygon and $Q(x)$ a solution of the differential equation, or if both are solutions, the same theorem evidently holds true, because then the function σ is identically zero, or else both ρ and σ vanish.

The polygons $P_n(x)$ all have a common interval. For take positive constants a and b such that the rectangle

$$0 \leq x - \xi \leq a \quad |y - \eta| \leq b$$

is entirely within R , and consequently has two constants m and k analogous to those above for R_1 . The portions of the polygons in the rectangle (6) all lie between the straight lines

$$y - \eta = \pm m(x - \xi)$$

since the slope of any side of any one of them is numerically less than m . It follows that each is certainly well defined and within the rectangle over an interval $\xi \leq x \leq a_1$, where a_1 is the smaller of a and b/m .

We shall now prove that the sequence of polynomials $P_n(x)$ converges uniformly, on the interval $\xi \leq x \leq \xi + a_1$, to a function $y(x)$ which has a continuous derivative and satisfies the differential equation (1). The curve $y = y(x)$ so defined is entirely within the region R .

For take $\varepsilon > 0$ arbitrarily, and l so small that

$$\frac{l}{k} |e^{ka_1} - 1| < \varepsilon$$

$$\text{Then} \quad |P_n(x) - P_n(x)| < \frac{l}{k} \{e^{ka_1} - 1\} < \varepsilon$$

provided that the intervals $\delta/2^{n'-1}$ and $\delta/2^{n-1}$ are each less than the constant μ corresponding to l . Hence the sequence $P_n(x)$ converges uniformly to a continuous function $y(x)$ on the interval $\xi \leq x \leq \xi + a_1$. The equations

$$P_n(x) = \eta + \int_{\xi}^x P_n'(x) dx = \eta + \int_{\xi}^x \{f(x, P_n) + \rho_n\} dx$$

hold for every n , and the sequences $\{f(x, P_n)\}$ and $\{\rho_n\}$ approach uniformly the limits $f\{x, y(x)\}$ and zero, respectively. Hence

$$y(x) = \eta + \int_{\xi}^x f\{x, y(x)\} dx$$

from which it follows by differentiation that $y(x)$ is a solution of the differential equation.

It is easy to show by means of the inequality (3) that there is only one continuous solution $y = y(x)$ of the differential equation (1) in the region R and passing through (ξ, η) . For suppose there were another, $Y(x)$, distinct from $y(x)$ at a value $x' > \xi$. There would then be a value $\xi_1 < x'$ at which $y(\xi_1) = Y(\xi_1)$, and such that the two solutions would be distinct throughout the interval $\xi_1 < x \leq x'$. In a neighbourhood of the point of intersection (ξ_1, η_1) interior to R , a relation

$$\left| \frac{d(Y - y)}{dx} \right| = |f(x, Y) - f(x, y)| < k|Y - y|$$

would be satisfied, and hence, from the inequality (3)

$$|Y - y| \leq 0$$

This contradicts the hypothesis that $y(x)$ and $Y(x)$ are distinct throughout the interval $\xi_1 < x \leq x_1'$.

It should be mentioned that Bendixson has shown* that if $y = \varphi(x, \xi, \eta)$, then $\varphi(x, \xi, \eta)$ and its derivative $\varphi_x(x, \xi, \eta)$ are continuous in all three of their arguments, and if the function $f(x, y)$ has continuous first derivatives with respect to x and y in the interior of the region R , then φ and φ_x have also continuous first derivatives with respect to all their arguments.

Perron has recently shown† that a solution of the equation

$$\frac{dy}{dx} = f(x, y)$$

exists and can be constructed whenever the function $f(x, y)$ is continuous over a region containing the initial point (x_0, y_0) .

EXAMPLE.

1. Find a solution of the equation $\frac{dy}{dx} = x^m y$ by the method of successive approximations, having given that $y = 1$ when $x = 0$. Prove that the method succeeds if $m > -1$.

2. The differential equation $\frac{dy}{dx} = \sqrt{y}$ possesses a solution

$$\begin{aligned} y(x) &= \frac{1}{4}(x - x_0 + 2\sqrt{y_0})^2 & x &\geq x_0 - 2\sqrt{y_0} \\ &= 0 & x &\leq x_0 - 2\sqrt{y_0} \end{aligned}$$

* *Bull. de la Soc. Math. de France*, Vol. 24 (1896), p. 220. See also G. A. Bliss, *loc. cit.*, p. 95.

† *Math. Ann.* Bd. 76 (1915), p. 471.

which satisfies the conditions

$$\begin{aligned} y &= y_0 & \text{when } x &= x_0 \\ y &= 0 & \text{when } x &= 0 \quad (\text{O. Perron.}) \end{aligned}$$

§ 79. *The integration of linear partial differential equations by means of infinite series.* In physical mathematics it is frequently necessary to find a solution of a harmonic linear partial differential equation

$$L_x(u) = M_t(u) \quad . \quad . \quad . \quad (1)$$

under the conditions that u satisfies certain supplementary equations or boundary conditions for all values of t within a certain range, and certain other supplementary conditions for all values of x within a certain interval. We shall find it convenient to call these two sets of supplementary conditions X and T respectively.

The method which has proved most successful in the treatment of problems of this kind is based on the fact that the sum of any number of particular solutions of a linear differential equation of the above type is generally a solution of the equation. Now, in the present case the partial differential equation possesses an infinite number of particular solutions of type

$$u = \alpha(x)\beta(t)$$

where $\alpha(x)$ and $\beta(t)$ satisfy the ordinary linear differential equations

$$L_x(\alpha) + \lambda\alpha = 0 \quad M_t(\beta) + \lambda\beta = 0$$

λ being an arbitrary constant. Let us suppose that a series of constants $\lambda_1, \lambda_2, \dots$ and a corresponding series of functions $\alpha_1(x), \beta_1(t); \alpha_2(x), \beta_2(t); \dots$ can be found such that each of the functions $\alpha_n(x)$ satisfies the conditions X , and each of the functions $\beta_n(t)$ satisfies all but one of the conditions T . Then, if $\alpha_n(x), \beta_n(t)$ satisfy the equations

$$L_x(\alpha) + \lambda_n\alpha = 0 \quad M_t(\beta) + \lambda_n\beta = 0 \quad . \quad . \quad (2)$$

respectively, the series

$$u = \sum_{n=1}^{\infty} A_n \alpha_n(x) \beta_n(t) \quad . \quad . \quad . \quad (3)$$

will satisfy the partial differential equation (1), the conditions X and all but one of the conditions T whatever may be the values of the constant coefficients A_n . The series (3) will represent the complete solution of the problem if the coefficients A_n can be chosen in a unique manner so that the last of the conditions T is satisfied, and so that the series converges in a suitable manner if m is infinite.

The last condition which has to be satisfied is generally of the form

$$u = f(x) \quad \text{when } t = 0 \quad \text{and } a < x < b \quad . \quad . \quad (4)$$

Putting $t = 0$ in equation (3), we see that if the series represents a continuous function of t , the coefficients A_n must be chosen so that

$$f(x) = \sum_{n=1}^m A_n \alpha_n(x) \beta_n(0) \quad (a < x < b) \quad . \quad . \quad (5)$$

If m is not infinite, it is only possible to determine the coefficients A_n when the function $f(x)$ is of a very restricted type, but when m is infinite, as is generally the case, it is often possible to expand a very general type of function in a series of the above form.

The coefficients A_n may frequently be determined with the aid of certain integral relations satisfied by the functions $\alpha_n(x)$. To illustrate this, let us consider the special case when

$$L_x(u) \equiv \frac{d}{dx} \left[p \frac{du}{dx} \right] + qu$$

$p, \frac{dp}{dx}$ and q being continuous functions of x . We may then deduce from the equations

$$L_x(\alpha_n) + \lambda_n \alpha_n = 0 \quad L_x(\alpha_r) + \lambda_r \alpha_r = 0$$

that
$$\frac{d}{dx} \left[p \left(\alpha_r \frac{d\alpha_n}{dx} - \alpha_n \frac{d\alpha_r}{dx} \right) \right] + (\lambda_n - \lambda_r) \alpha_n \alpha_r = 0$$

Now, it frequently happens that in virtue of the conditions X the quantity $p \left(\alpha_r \frac{d\alpha_n}{dx} - \alpha_n \frac{d\alpha_r}{dx} \right)$ is zero when $x = a$ and when $x = b$. In this case we can generally conclude that

$$\int_a^b \alpha_n(x) \alpha_r(x) dx = 0 \quad (\lambda_n \neq \lambda_r)$$

When $r = n$, the integral is different from zero and usually finite ; consequently, we may write

$$\int_a^b [\alpha_n(x)]^2 dx = \mu_n$$

Multiplying the series (5) by $\alpha_n(x)$ and integrating term by term, we see that when this last process is justifiable,

$$\int_a^b f(x) \alpha_n(x) dx = A_n \mu_n \beta_n(0)$$

and the coefficient A_n is thereby determined.

Mathematicians have spent a large amount of time in perfecting this method and in justifying the various steps. The points that have required so much investigation will now be enumerated.

1°. The differential equation $L_x(\alpha) + \lambda \alpha = 0$ has to be studied in connection with the conditions X ; the possible values of λ have to be ascertained and certain questions regarding the distribution of these values have to be answered. It is important to know, for instance, if the values of λ are infinite in number, whether they are isolated or not, whether they are real or imaginary. For a dis-

cussion of these questions the reader is referred to Prof. Bôcher's article,* "Boundary problems in one dimension."

In the case of Legendre's equation

$$\frac{d}{dx} \left[(1 - x^2) \frac{d\alpha}{dx} \right] + \lambda \alpha = 0$$

if the conditions X state that α is to be finite when $x = 1$ and $x = -1$, the constant λ must have a value such as $s(s+1)$, where s is an integer. These values are all real and infinite in number.

2°. It is of some interest to ascertain the different possible types of conditions X for which $p \left(\alpha_r \frac{d\alpha_n}{dx} - \alpha_n \frac{d\alpha_r}{dx} \right)$ is zero at the limits a and b . A useful discussion of this question is given in a paper by D. Hilbert.†

In the case of Legendre's equation, the conditions stated above may be used, provided $a = -1$, $b = +1$.

3°. The nature of the convergence of a series of type

$$\sum c_n \alpha_n(x)$$

must be ascertained. The question of the possibility of expanding a given function $f(x)$ in a series of this type must be answered, and finally the series (3) must be studied with the object of finding if it represents a continuous function of t for $t = 0$. These questions are too difficult to be studied here in their full generality, so we shall limit the present discussion to that of a simple case which will illustrate the method. For the more elaborate studies the reader is referred to some of the original memoirs.‡

Let us consider the partial differential equation

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad (0 < x < l)$$

and the supplementary conditions

$$(X) \quad v = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = l \quad (t > 0)$$

$$(T) \quad v \text{ is finite for } t = +\infty$$

$$v = f(x) \quad \text{for} \quad t = 0 \quad (0 < x < l)$$

* *Proceedings of the 5th International Congress of Mathematicians*, Cambridge (1912), Vol. 1, p. 162. See also G. D. Birkhoff, *Trans. Amer. Math. Soc.*, Vol. 10, 1909, p. 259.

† *Göttinger Nachrichten* (1904). See also G. D. Birkhoff, *l.c.*

‡ See, for instance, A. Kneser, *Math. Ann.*, Vol. 58 (1903), p. 81; Vol. 60 (1905), p. 402; Vol. 63 (1907), p. 477; D. Hilbert, *loc. cit.*; M. Mason, *Trans. Amer. Math. Soc.*, Vol. 8 (1907), p. 431; A. C. Dixon, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, p. 83; Vol. 4, p. 411; E. W. Hobson, *Ibid.*, Vol. 6, p. 349; Vol. 7, pp. 24, 359; J. Mercer, *Phil. Trans.*, A. Vol. 211 (1912), p. 111; A. C. Dixon, *Ibid.*, p. 411; Haar, *Math. Ann.*, Vol. 69 (1910), p. 331; Vol. 71 (1911), p. 38.

The equation $\frac{d^2v}{dx^2} + \lambda v = 0$

possesses a solution $v = A \sin(x\sqrt{\lambda}) + B \cos(x\sqrt{\lambda})$, which satisfies the conditions X if $l\sqrt{\lambda} = n\pi$, where n is an integer; for then it is sufficient to take $B = 0$.

The equation $\frac{dv}{dt} + \lambda v = 0$

possesses a solution $v = Ce^{-\lambda t}$, which satisfies the first of conditions T if the real part of λ is not negative. The value $\lambda = n^2\pi^2/l^2$ certainly satisfies this requirement.

We must now consider the infinite series

$$v = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

If this represents a continuous function of t for $t = 0$, it will satisfy the second of conditions T if

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

Writing $\alpha_n(x) = \sin \frac{n\pi x}{l}$ $\alpha_r(x) = \sin \frac{r\pi x}{l}$

we see that the condition

$$\alpha_r \frac{d\alpha_n}{dx} - \alpha_n \frac{d\alpha_r}{dx} = 0 \quad \text{for } x = 0 \quad \text{and } x = l$$

is satisfied; hence the coefficients in the expansion may be determined with the aid of the formula

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{r\pi x}{l} dx = 0 \quad (n \neq r)$$

We also find by direct integration that

$$\int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{1}{2}l$$

Now Fourier's theorem* tells us that if $f(x)$ is a single-valued function of x in the interval $0 \leq x \leq l$, and is such that the integral

$$\int_0^l f(x) dx$$

is convergent, and, moreover, if the number of discontinuities and maxima and minima of the function $f(x)$ in the interval $0 \leq x \leq l$ is finite, then the series

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l},$$

* See Carslaw's *Fourier Series and Integrals*, p. 133.

where

$$A_n = \frac{2}{l} \int_0^l f(\xi) \sin \frac{n\pi\xi}{l} d\xi,$$

converges at all points of the interval $0 < x < l$, with the possible exception of the points of discontinuity of $f(x)$. The series represents the function $f(x)$ for any value of x for which $f(x)$ is continuous, while if $f(x)$ has a finite discontinuity at x , the series converges to the value

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2} [f(x + \epsilon) + f(x - \epsilon)]$$

Let us assume that $f(x)$ satisfies the conditions of this theorem ; * then, since the series is convergent, we may conclude that each of its terms, at any rate after some finite value of n , is numerically less than some finite positive quantity M . If, moreover, we exclude the points of discontinuity of $f(x)$ at which the series does not converge by small intervals, we may choose M so that it is independent of x , and make the finite value of n to which we have just referred also independent of x . We now have

$$\left| A_n \sin \frac{n\pi x}{l} \right| < M \quad \text{for } 0 \leq x \leq l$$

It follows from this inequality that if $0 < t_0 < t$

$$\left| n^{2s} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}} \right| < M n^{2s} e^{-\frac{n^2\pi^2 t_0}{l^2}}$$

Now the series

$$\sum_{n=1}^{\infty} M n^{2s} e^{-\frac{n^2\pi^2 t_0}{l^2}}$$

converges and is independent of x and t ; hence it follows that the series

$$\sum_{n=1}^{\infty} n^{2s} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}}$$

converges absolutely and uniformly for any range of x which does not include any of the excluded intervals and for any positive interval of t .

Putting $s = 0, 1$ in succession, we see that the series

$$v = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}} - \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} A_n n^2 \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}}$$

converge uniformly ; hence the second series may be derived from the first either by differentiating it term by term with regard to t or by differentiating it twice term by term with regard to x .† This

* These are usually called Dirichlet's conditions.

† The cosine series which is derived by one differentiation is convergent, because it can be derived from a uniformly convergent series (viz. the second series) by integration with regard to x . We may prove by an application of the preceding method that the cosine series is uniformly convergent.

indicates that the function v represented by the infinite series satisfies the partial differential equation $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$, except possibly in the excluded intervals.

In the problems of mathematical physics the function $f(x)$ does not generally become infinite, and so we shall assume that there are no excluded intervals.

We have now to prove that as $t \rightarrow 0$ our function v tends to the limit $f(x)$. To do this, let us write

$$a_n = A_n \sin \frac{n\pi x}{l} \quad c_n = e^{-\frac{n^2 \pi^2 t}{l^2}}$$

$${}_p R_n(t) = a_n c_n + a_{n+1} c_{n+1} + \dots a_{n+p+1} c_{n+p+1} \\ {}_p r_n = a_n + a_{n+1} + \dots a_{n+p-1}$$

Then it is easy to see that

$${}_p R_n(t) = {}_1 r_n (c_n - c_{n+1}) + {}_2 r_n (c_{n+1} - c_{n+2}) \\ + \dots {}_{p-1} r_n (c_{n+p-2} - c_{n+p-1}) + {}_p r_n c_{n+p-1}$$

But $\sum_{n=1}^{\infty} a_n$ is convergent and $c_n > c_{n+1} > c_{n+2} > c_{n+p-1}$; therefore

$$|{}_p r_n| < \varepsilon \quad \text{for } \begin{matrix} n \geq v \\ p = 1, 2, \dots \end{matrix}$$

$$\text{and } |{}_p R_n(t)| < \varepsilon [(c_n - c_{n+1}) + (c_{n+1} - c_{n+2}) \\ + \dots (c_{n+p-2} - c_{n+p-1}) + c_{n+p-1}] \\ < \varepsilon c_n < \varepsilon \quad \text{for } \begin{matrix} n \geq v \\ p = 1, 2, \dots \end{matrix}$$

Hence it follows that the series $\sum a_n c_n$ is uniformly convergent in any interval for which $t > 0$, and the uniformity of the convergence extends up to and includes the point $t = 0$. Hence v is a continuous function of t for $0 \leq t$, and so we may conclude that

$$\lim_{t \rightarrow 0} v(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = f(x)$$

The function v defined by the infinite series thus satisfies all the conditions of the problem.

The general method of solving harmonic partial differential equations which has just been described may be extended to partial differential equations of the form

$$L_x(u) + M_y(u) + N_z(u) = 0$$

where L_x , M_y , N_z are linear differential operators. The particular solutions from which the required solution is built up are now of the type

$$u = A(x)B(y)C(z)$$

where

$$L_x(A) + \lambda A = 0 \quad M_y(B) + \mu B = 0 \quad N_z(C) + \nu C = 0$$

and

$$\lambda + \mu + \nu = 0$$

This method has already been used in the case of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and can also be applied when the equation is transformed to polar coordinates.

We may, in fact, generalise the particular solutions of § 66 by writing

$$u = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} r^n \cos(m\phi + \epsilon_{m,n}) A_{m,n} P_n^m(\cos \theta)$$

where $\epsilon_{m,n}$ and $A_{m,n}$ are constants at our disposal. In many cases the conditions of the physical problem indicate that $A_{m,n}$ is zero either when n is negative or when it is not negative.

EXAMPLES.

1. Prove that the equation

$$\frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial V}{\partial \mu} \right] = \frac{\partial}{\partial \nu} \left[(1 - \nu^2) \frac{\partial V}{\partial \nu} \right]$$

is satisfied by a series of type

$$V = \sum A_n P_n(\mu) P_n(\nu)$$

where $P_n(\mu)$ is Legendre's polynomial. Show also that if $V = f(\mu)$ when $\nu = 1$, the coefficients A_n may be derived with the aid of the formula

$$\int_{-1}^{+1} P_m(\mu) P_n(\mu) d\mu = 0 \quad (m \neq n)$$

and deduce this formula from Legendre's equation.

2. Prove that the preceding differential equation is satisfied by

$$V = (\mu + \nu) P_n \left(\frac{1 + \mu\nu}{\mu + \nu} \right)$$

Hence obtain an expansion for this function in a series of Legendre polynomials.

3. If $H_n(x) = F(a + n; -n; c; x)$, where n is zero or a positive integer and c is positive, we have for $a > c - 1$

$$\int_0^1 H_m(x) H_n(x) x^c (1-x)^{a-c} dx = 0 \quad (m \neq n)$$

When $m = n$ the value of the integral is

$$\frac{1}{a + 2n} \frac{\Gamma(n+1) [\Gamma(c)]^2 \Gamma(a+n-c+1)}{\Gamma(a+n) \Gamma(c+n)} \quad (\text{Jacobi.})$$

CHAPTER X

THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS

§ 80. *Laplace's transformation.* The method which will now be explained can often be employed with advantage when the differential equation we wish to solve has the form

$$L_x(u) \equiv \sum_{m=0}^p \sum_{n=0}^q a_{m,n} x^n \frac{d^m u}{dx^m} = 0 \quad . \quad . \quad . \quad (1)$$

and is particularly successful when $q = 1$.

The success of the method depends on the possibility of finding an integrating factor of the expression

$$M_t(v) \equiv \sum_{m=0}^p \sum_{n=0}^q a_{m,n} t^m \frac{d^n v}{dt^n}$$

which is obtained from the left-hand side of (1) by replacing x^n by $\frac{d^n v}{dt^n}$ and $\frac{d^m u}{dx^m}$ by t^m . If $q = 1$, an integrating factor can be found at once by the method of § 7. In the general case an integrating factor is a solution of the equation $\bar{M}_t(w) = 0$, adjoint to $M_s(v) = 0$. This equation is defined as in § 18 by the identity

$$w M_t(v) - v \bar{M}_t(w) \equiv \frac{d}{dt} R(v, w)$$

and in the present case we have

$$\bar{M}_t(w) \equiv \sum_{m=0}^p \sum_{n=0}^q (-1)^n a_{m,n} \frac{d^n}{dt^n} (t^m w)$$

$$R(v, w) \equiv \sum_{m=0}^p \sum_{n=0}^q a_{m,n} F_n(v, t^m w)$$

where

$$z \frac{d^n v}{dt^n} - v \frac{d^n z}{dt^n} (-1)^n \equiv \frac{d}{dt} F_n(v, z)$$

$$F_n(v, z) = z \frac{d^{n-1} v}{dt^{n-1}} - \frac{dz}{dt} \frac{d^{n-2} v}{dt^{n-2}} + \frac{d^2 z}{dt^2} \frac{d^{n-3} v}{dt^{n-3}} - \dots$$

After these preliminaries let us endeavour to satisfy the differential equation by a definite integral of the type first used by Euler,

$$u = \int e^{xt} w(t) dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the path of integration is at present unspecified.

Assuming that the definite integral can be differentiated under the integral sign a suitable number of times by the rule of Leibnitz, we have

$$L_x(u) = \int L_x(e^{xt}) w(t) dt$$

It should now be observed that we have identically

$$L_x(e^{xt}) \equiv M_t(e^{xt})$$

hence

$$L_x(u) = \int M_t(e^{xt}) w(t) dt$$

Now let $w(t)$ be chosen so that $\bar{M}_t(w) = 0$; then the expression under the integral sign is an exact differential, and can be written in the form

$$\frac{d}{dt} R(e^{xt}, w)$$

If, then, the limits of the integral (2) are chosen so that the quantity R vanishes at both limits or takes the same value at both limits, the definite integral will represent a solution of the differential equation (1).

The equation $\bar{M}_t(w) = 0$ is called the *Laplace transformed equation* of $L_x(u) = 0$. The relation between the two equations is such that if a solution of one can be obtained, then a solution of the other can generally be derived from it by Laplace's method. Petzval * has shown, in fact, that if e^{-xt} is used in place of e^{xt} , then $L_x(u) = 0$ is the Laplace transformed equation of $\bar{M}_t(w) = 0$.

To verify this we have simply to prove that

$$\bar{L}_x(e^{-xt}) = \bar{M}_t(e^{-xt})$$

Now
$$\bar{L}_x(u) \equiv \sum \sum (-1)^m a_{m,n} \frac{d^m}{dx^m} (x^n u)$$

hence the preceding equation holds if

$$(-1)^m \frac{d^m}{dx^m} (x^n e^{-xt}) \equiv (-1)^n \frac{d^n}{dt^n} (t^m e^{-xt})$$

and this can be shown to be an identity by using Leibnitz's theorem.

It follows from Petzval's result, that if the first equation can be satisfied by an integral of the form

$$u(x) = \int_{\tau_1}^{\tau_2} e^{xt} w(t) dt$$

* *Integration der linearen Differentialgleichungen*, p. 472.

the equation $\bar{M}_t(w) = 0$ can often be satisfied by an integral of the form

$$w(t) = \int_{t_1}^{t_2} e^{-xt} u(x) dx$$

To illustrate the use of Laplace's method, let us consider the equation

$$L_x(u) \equiv x \frac{d^2 u}{dx^2} + (p + q + x) \frac{du}{dx} + pu = 0 \quad . \quad . \quad (3)$$

In this case

$$M_t(v) \equiv t(t+1) \frac{dv}{dt} + [p + (p+q)t] v$$

$$\bar{M}_t(w) \equiv t(t+1) \frac{dw}{dt} - [p-1 + (p+q-2)t] w$$

hence we may take $w = t^{p-1}(1+t)^{q-1}$

and then $wM_t(v) = \frac{d}{dt}[t^p(1+t)^q v]$

The quantity $t^p(1+t)^q e^{xt}$ vanishes when $t = 0$ and when $t = -1$, provided $p > 0$, $q > 0$; it also vanishes when $t = -\infty$ if $x > 0$. Writing $-t$ in place of t , we obtain the two solutions

$$u_1(x) = \int_0^1 e^{-xt} t^{p-1} (1-t)^{q-1} dt$$

$$u_2(x) = \int_0^\infty e^{-xt} t^{p-1} (1-t)^{q-1} dt \quad p > 0, q > 0, x > 0$$

Sometimes it is convenient to use a definite integral of the type

$$\int \sin(xt + \epsilon) w(t) dt$$

in place of an integral containing the exponential function; in this case we seek a relation of the form

$$L_x[\sin(xt + \epsilon)] = M_t[\sin(xt + \epsilon_1)]$$

For instance, if

$$L_x(u) \equiv x \frac{d^2 u}{dx^2} + (2\nu + 1) \frac{du}{dx} + xu = 0 \quad . \quad . \quad (4)$$

we have $M_t(v) \equiv (t^2 - 1) \frac{dv}{dt} + (2\nu + 1) tv$

$$\epsilon_1 = \frac{\pi}{2} + \epsilon$$

An integrating factor of $M_t(v)$ is easily found to be

$$w(t) = (t^2 - 1)^{\nu-1}$$

and we have

$$wM_t[\cos(xt + \epsilon)] = \frac{d}{dt}[(t^2 - 1)^{\nu+1} \cos(xt + \epsilon)]$$

If $\nu > -\frac{1}{2}$, the quantity within square brackets is zero when $t = \pm 1$; hence one solution of the differential equation (4) is given by

$$u = \int_{-1}^{+1} \sin(xt + \epsilon)(1 - t^2)^{\nu-\frac{1}{2}} dt$$

where ϵ is an arbitrary constant. This solution is of the form

$$u = \sin \epsilon \int_{-1}^{+1} \cos xt (1 - t^2)^{\nu-\frac{1}{2}} dt$$

and is finite when $x = 0$.

Another definite integral satisfying the differential equation may be obtained in the following way. Let the identity

$$L_x[w \sin xt] = \frac{d}{dt}[(t^2 - 1)^{\nu+\frac{1}{2}} \cos xt]$$

be integrated twice with regard to x between 0 and x ; then, if $V(x) = w(t) \sin xt$, we get

$$\left. \begin{aligned} xV(x) + (2\nu - 1) \int_0^x V(x_1) dx_1 + \int_0^x dx_1 \int_0^{x_1} x_2 V(x_2) dx_2 \\ = \frac{d}{dt} \left[(t^2 - 1)^{\nu+\frac{1}{2}} \frac{1 - \cos xt}{t^2} \right] \end{aligned} \right\} \quad (5)$$

Now, if $-\frac{1}{2} < \nu < \frac{1}{2}$, the quantity within square brackets is zero when $t = 1$ and also when $t = \infty$; hence it follows that the function

$$u = \int_1^\infty \sin xt (t^2 - 1)^{\nu-\frac{1}{2}} dt \quad \dots \dots \dots (6)$$

satisfies the equation

$$xu(x) + (2\nu - 1) \int_0^x u(x_1) dx_1 + \int_0^x dx_1 \int_0^{x_1} x_2 u(x_2) dx_2 = 0 \quad (7)$$

Differentiating twice, we find that u satisfies the differential equation (4). It should be noticed that in this case it is not permissible to differentiate the integral under the integral sign, but the integrations under the integral sign which are needed when the expression (6) is substituted in (7) are quite justifiable. It is now easy to verify with the aid of (5) that the definite integral (6) satisfies equation (7).

§ 81. *Equations of Pfaff's type.* A differential equation which can be written in the symbolical form

$$\left[x^n F\left(x \frac{d}{dx}\right) + G\left(x \frac{d}{dx}\right) \right] u = 0 \quad \dots \dots \dots (8)$$

is said to be of Pfaff's type, in honour of Pfaff's investigations; the equation appears to have been first discussed by Euler.

A solution of a differential equation of this type can often be expressed as a definite integral of the form

$$u = \int_{\tau_0}^{\tau_1} K(xt)w(t) dt \quad \dots \dots \dots (9)$$

The method depends on the fact that the function $K(xt) = K(z)$ satisfies the partial differential equation

$$\left[x^n F\left(x \frac{d}{dx}\right) + G\left(x \frac{d}{dx}\right) \right] K = \left[G\left(t \frac{d}{dt}\right) + t^{-n} H\left(t \frac{d}{dt}\right) \right] K$$

if
$$\left[z^n F\left(z \frac{d}{dz}\right) - H\left(z \frac{d}{dz}\right) \right] K = 0$$

Now

$$\left[x^n F\left(x \frac{d}{dx}\right) + G\left(x \frac{d}{dx}\right) \right] u = \int w(t) \left[G\left(t \frac{d}{dt}\right) + t^{-n} H\left(t \frac{d}{dt}\right) \right] K$$

hence the function $w(t)$ should be chosen so that the quantity under the integral sign is an exact differential; in other words, w must be a solution of the differential equation adjoint to

$$\left[G\left(t \frac{d}{dt}\right) + t^{-n} H\left(t \frac{d}{dt}\right) \right] v = 0$$

In the case of the hypergeometric equation

$$x(x-1) \frac{d^2 u}{dx^2} + \{(a+b+1)x - c\} \frac{du}{dx} + abu = 0. \quad (10)$$

we choose $K(xt)$ so that

$$\begin{aligned} x^2(x-1) \frac{\partial^2 K}{\partial x^2} + \{(a+b+1)x^2 - cx\} \frac{\partial K}{\partial x} + abxK \\ = t(1-t) \frac{\partial^2 K}{\partial x^2} + (d-ct) \frac{\partial K}{\partial t} \end{aligned}$$

This means that

$$z(z-1) \frac{d^2 K}{dz^2} + \{(a+b+1)z - d\} \frac{dK}{dz} + abK = 0$$

hence we may take $K(xt) = F(a, b, d, xt)$

An integrating factor $w(t)$ is given by $w = t^{d-1}(1-t)^{c-d-1}$, and the limits of the definite integral (9) must be chosen so that

$$\left[t^d (1-t)^{c-d} \frac{dK}{dt} \right]_{\tau_0}^{\tau_1} = 0$$

If $d > 0$, $c > d$, we may take $\tau_0 = 0$, $\tau_1 = 1$, and we may conclude that the integral

$$\int_0^1 t^{d-1} (1-t)^{c-d-1} F(a, b, d, xt) dt$$

satisfies the hypergeometric equation (10). We have, in fact, the formula

$$\int_0^1 t^{d-1} (1-t)^{c-d-1} F(a, b, d, xt) dt = \frac{\Gamma(d)\Gamma(c-d)}{\Gamma(c)} F(a, b, c, x)$$

which may be verified by expanding both sides in ascending powers of x . In particular, if $d = b$, we have

$$F(a, b, b, xt) = (1 - xt)^{-a}$$

and we obtain Euler's formula

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, c, x)$$

which holds when $|x| < 1$ and the real parts of b and $c - b$ are positive.

§ 82. *Euler's transformation.* Euler used definite integrals of type

$$u(x) = \int \frac{w(t) dt}{(t-x)^{\nu+1}} \dots \dots \dots (1)$$

to represent solutions of linear differential equations, and the transformation which enables us to pass from a differential equation satisfied by $u(x)$ to the corresponding differential equation satisfied by $w(t)$ is known as *Euler's transformation*, although the analysis of the transformation is due chiefly to Heine,* Pincherle,† Jordan,‡ Pochhammer§ and other writers.||

Let us suppose that the given differential equation $L_x(u) = 0$ can be written in the form

$$\begin{aligned} L_x(u) \equiv P(x) \frac{d^n u}{dx^n} - \alpha P'(x) \frac{d^{n-1} u}{dx^{n-1}} + \frac{\alpha(\alpha+1)}{1 \cdot 2} P''(x) \frac{d^{n-2} u}{dx^{n-2}} - \dots \\ - Q(x) \frac{d^{n-1} u}{dx^{n-1}} + (\alpha+1) Q'(x) \frac{d^{n-2} u}{dx^{n-2}} - \dots \\ + R(x) \frac{d^{n-2} u}{dx^{n-2}} - \dots \end{aligned}$$

where $P(x)$, $Q(x)$, $R(x)$, ... are polynomials in x of degrees n , $n-1$, $n-2$, ... respectively.

Writing $\nu = -\alpha - n$ and substituting the expression (1) in the differential equation, we find that, since by Taylor's theorem

$$P(x) + (t-x)P'(x) + \frac{(t-x)^2}{1 \cdot 2} P''(x) + \dots = P(t)$$

$$\begin{aligned} L_x(u) &= (-1)^n (\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1) \alpha \\ &\times \int \left[(t-x)^{\alpha-1} P(t) + \frac{1}{\alpha} (t-x)^\alpha Q(t) + \frac{(t-x)^{\alpha+1}}{\alpha(\alpha+1)} R(t) + \dots \right] w(t) dt \end{aligned}$$

* *Crelle's Journal*, Bd. 60 (1862), p. 252; Bd. 61 (1863), p. 356; Bd. 62 (1863), p. 110; *Handbuch der Kugelfunktionen*, Bd. 1, 2nd ed. Berlin (1881), p. 466.

† *Acta Mathematica*, t. 10 (1886), p. 153; *Crelle's Journal*, Bd. 119 (1898), p. 347.

‡ *Cours d'analyse*, t. 3, Paris (1887), p. 241.

§ *Math. Ann.*, Bd. 35 (1890), pp. 470, 495; Bd. 37, p. 500; *Crelle's Journal*, Bd. 71 (1870), p. 316; Bd. 73 (1871), p. 69.

|| See especially E. W. Hobson, *Phil. Trans.*, A. Vol. 187, p. 493.

If there are $p + 1$ terms in the expression under the integral sign, we may write the preceding equation in the form

$$L_x(u) = (-1)^n(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + p) \\ \times \int M_t[(t - x)^{\alpha+p-1}] w(t) dt$$

where $M_t(v)$ denotes the expression

$$P(t) \frac{d^p v}{dt^p} + Q(t) \frac{d^{p-1} v}{dt^{p-1}} + R(t) \frac{d^{p-2} v}{dt^{p-2}} + \dots$$

The equation adjoint to $M_t(v)$ is called the *Euler transformed equation* of $L_x(u)$. It should be noticed that we have the identity

$$L_x[(t - x)^{\alpha+n-1}] \\ = (-1)^n(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)\alpha M_t[(t - x)^{\alpha+p-1}]$$

The method is particularly successful when $p = 1$, for then $M_t(v)$ is simply

$$P(t) \frac{dv}{dt} + Q(t)v$$

and the function $w(t)$ must be chosen so that it is an integrating factor of this expression.

To illustrate the method, let us consider Legendre's equation

$$L_x(u) \equiv (1 - x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} + n(n + 1)u = 0$$

In this case we have $P(x) = 1 - x^2$,

$$\alpha P'(x) + Q(x) = 2x, \quad \frac{\alpha(\alpha + 1)}{1 \cdot 2} P''(x) + (\alpha + 1)Q'(x) = n(n + 1),$$

$$\therefore Q(x) = 2(1 + \alpha)x, \quad -\alpha(\alpha + 1) + 2(\alpha + 1)^2 = n(n + 1).$$

The last equation is satisfied by either $\alpha = n - 1$ or $\alpha = -n - 2$.

Now, an integrating factor of the expression

$$(1 - t^2) \frac{dv}{dt} + 2(1 + \alpha)t \cdot v$$

is given by the equation

$$\frac{d}{dt}[w(1 - t^2)] = 2(1 + \alpha)t \cdot w$$

Hence $w = (1 - t^2)^{-\alpha-2}$, and we have

$$M_t[(t - x)^\alpha] \cdot (1 - t^2)^{-\alpha-2} = \frac{d}{dt}[(t - x)^\alpha (1 - t^2)^{-\alpha-1}]$$

The path of integration must be chosen so that the quantity within the last square bracket has the same value at the two ends of the path.

When $\alpha = -n - 2$ and $n + 1$ is positive, the conditions may be satisfied by taking $t = -1$ and $t = +1$ as the limits of inte-

gration ; hence, if x does not belong to the interval $(-1 < x < 1)$ we may conclude that the integral

$$u(x) = \int_{-1}^{+1} \frac{(1 - t^2)^n dt}{(t - x)^{n+1}}$$

satisfies Legendre's equation. This integral may be identified with the function $2^{n+1}Q_n(x)$.

The conditions may also be satisfied by taking the path of integration to be a closed contour in the plane of the complex variable t . If this contour encloses the points $t = 1$ and $t = x$, but not the point $t = -1$, the expression

$$\frac{(1 - t^2)^{n+1}}{(t - x)^{n+2}}$$

will return to its initial value after describing the contour. To see this we notice that the arguments of $t - 1$, $t - x$ increase by 2π after one description of the closed contour, while the argument of $t + 1$ returns to its initial value. The argument of the expression $\frac{(1 - t^2)^{n+1}}{(t - x)^{n+2}}$ thus changes by

$$(n + 1)2\pi - (n + 2)2\pi = -2\pi.$$

and its modulus is unaltered. Since $e^{-2\pi i} = 1$, we see that the expression returns to its initial value.

$$\text{The definite integral } u = \frac{1}{2\pi i} \int_c \frac{(1 - t^2)^n dt}{(t - x)^{n+1}}$$

is thus a solution of Legendre's equation whether n is integral or not ; it can be identified with the function $2^n P_n(x)$.

Linear differential equations have been solved recently with great success by means of definite integrals of type

$$\int x^s w(s) ds$$

in which $w(s)$ is the ratio of two products of Γ functions of type $\Gamma(a \pm s)$. An account of this method, which is due to Pincherle, is given in a paper by E. W. Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. VI. p. 141.

EXAMPLES.

1. Prove that the differential equation

$$\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \left[1 - \frac{n(n+1)}{x^2} \right] y = 0$$

is satisfied by a definite integral of type

$$y = \int e^{tx} P_n(t) dt$$

where $P_n(t)$ is a solution of Legendre's equation. Hence deduce Rayleigh's formula

$$\sqrt{2\pi i} x^{-1/2} J_{n+1/2}(x) = \int_{-1}^{+1} e^{ix\mu} P_n(\mu) d\mu$$

2. Prove that the differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right) y = 0$$

is satisfied by
$$y = \int_0^\infty e^{-x \cosh \phi} \cosh n\phi d\phi$$

3. Prove that the integral

$$y = \int_{-1}^{+1} \frac{P_n(\mu) d\mu}{x - \mu}$$

satisfies Legendre's equation.

4. Prove that the equation

$$\frac{d^2 y}{dx^2} + \frac{3}{x} \frac{dy}{dx} + \left(4 - \frac{4n^2 - 1}{x^2}\right) \frac{dy}{dx} + \frac{4}{x} y = 0$$

is satisfied by
$$y = \int_0^\infty e^{-xt} P_n(1 + \frac{1}{2}t^2) dt$$

when $n = m + \frac{1}{2}$ and m is a positive integer.

5. Prove that
$$\int_0^\infty e^{-xt} J_{n+1/2}^2(x) dx = \frac{1}{\pi} Q_n(1 + \frac{1}{2}t^2). \quad (\text{H. M. Macdonald.})$$

6. If $\mathfrak{D}$ denotes the operator $x \frac{d}{dx}$

$$[xF(\mathfrak{D}) + G(\mathfrak{D})]x^s w(s) = [F(s)x^{s+1} + G(s)x^s]w(s)$$

Prove that the quantity on the right-hand side of this equation is a perfect difference $\Phi(s+1) - \Phi(s)$ if $w(s)$ satisfies the difference equation

$$G(s)w(s) + F(s-1)w(s-1) = 0$$

7. Prove that equation (10) of § 81 is satisfied by an integral of type

$$u = \int \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-x^s) ds$$

when $|\arg(-x)| < \pi$ and the contour of integration is parallel to the imaginary axis in the complex s -plane, with loops, if necessary, to ensure that the points, 0, 1, 2, ... are to the right, and the points $-a$, $-a-1$, $-a-2$, ..., $-b$, $-b-1$, $-b-2$, ... are to the left of the contour.

§ 83. *The solution of partial differential equations by means of definite integrals.* There are two types of definite integrals which are used to represent solutions of linear partial differential equations. In the first type the integrand is itself a solution of the partial differential equation for all values of the parameters with respect to which the integration takes place. In the second type of definite integral the integrand is not a solution of the partial differential equation.

To obtain a solution of a given linear partial differential equation in the form of an integral of the first type we must first of all find a particular solution involving one or a number of arbitrary parameters. If the differential equation is of the form

$$F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) V = 0 \quad \dots \quad (1)$$

a particular solution may be found by writing $V = e^{\xi x + \eta y}$.

The differential equation is then satisfied if $F(\xi, \eta) = 0$. Let us suppose that the coordinates of points on this curve $F = 0$ can be expressed as functions of a parameter s ; then $V = e^{x\xi(s) + y\eta(s)}$ is a particular solution involving a parameter s , and this solution may be generalised by writing

$$V = \int_{s_1}^{s_2} e^{x\xi(s) + y\eta(s)} \varphi(s) ds \quad \dots \quad (2)$$

where s_1, s_2 are arbitrary constants and $\varphi(s)$ is an arbitrary function subject to certain limitations.

In the case of the equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial y}$, we may take

$$\eta(s) = -s^2 \quad \xi(s) = is \quad s_1 = -\infty \quad s_2 = +\infty$$

thus when $y > 0$, we have the solution

$$V = \int_{-\infty}^{\infty} e^{-ys^2 + isx} \varphi(s) ds \quad \dots \quad (3)$$

In particular, when $\varphi(s) = 1$, we have

$$V = \int_{-\infty}^{\infty} e^{-y\left(s - \frac{ix}{2y}\right)^2 - \frac{x^2}{4y}} ds = e^{-\frac{x^2}{4y}} y^{-\frac{1}{2}} \int_{-\infty - ia}^{\infty - ia} e^{-z^2} dz$$

where
$$z = \sqrt{y} \left(s - \frac{ix}{2y} \right) \quad \alpha = \frac{x}{2\sqrt{y}}$$

Now it is easy to prove by Cauchy's theorem that

$$\int_{-\infty - ia}^{\infty - ia} e^{-z^2} dz = \int_{-\infty}^{\infty} e^{-z^2} dz = A, \text{ say}$$

Hence
$$V = Ay^{-\frac{1}{2}} e^{-\frac{x^2}{4y}}$$

The particular solution $V = y^{-\frac{1}{2}} e^{-\frac{x^2}{4y}}$ is called the *fundamental solution* of the differential equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial y}$. We may generalise it by writing $x - a$ in place of x , and integrating with regard to a after introducing an arbitrary function of a as a factor. Thus we obtain the solution

$$V = y^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{4y}} \chi(a) da \quad (y > 0) \quad \dots \quad (4)$$

Transforming this integral by the substitution $a = x + 2u\sqrt{y}$, we obtain a solution in the form of a definite integral of the second type, viz.

$$V = 2 \int_{-\infty}^{\infty} \chi(x + 2u\sqrt{y}) e^{-u^2} du$$

If the function $\chi(x)$ is such that the definite integral represents a continuous function of y as $y \rightarrow 0$, the solution V satisfies the condition

$$V = 2\chi(x) \int_{-\infty}^{\infty} e^{-u^2} du = 2A\chi(x) \quad \text{for } y = 0$$

It is shown in books on definite integrals that $A = \sqrt{\pi}$; hence a solution which satisfies the condition $V = \chi(x)$ when $y = 0$ is given by Laplace's formula

$$V = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \chi(x + 2u\sqrt{y}) e^{-u^2} du \quad \dots \quad (5)$$

The same problem may be solved in another way with the aid of formula (3). If we suppose that the function $\phi(s)$ is of such a nature that the definite integral represents a continuous function of y for $y = 0$, the function $\phi(s)$ must satisfy the equation

$$\chi(x) = \int_{-\infty}^{\infty} e^{ixs} \phi(s) ds \quad \dots \quad (6)$$

If the function $\chi(s)$ satisfies Dirichlet's conditions for an unlimited real interval and $\int_{-\infty}^{\infty} |\chi(x)| dx$ is convergent, the function $\phi(s)$ may be deduced from the above equation with the aid of Fourier's inversion formula *

$$\phi(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} \chi(x) dx \quad \dots \quad (7)$$

and then the solution of the problem is expressed by means of (3).

This method, which depends on the use of Fourier's formula and its generalisations, is of frequent application.

In the case of a harmonic linear partial differential equation

$$L_x(u) = M_t(u) \quad \dots \quad (8)$$

definite integral solutions may be found by generalising the solution of type

$$u = \alpha_v(x) \beta_v(t)$$

where $\alpha_v(x)$, $\beta_v(t)$ satisfy the equations

$$L_x(\alpha) + v\alpha = 0 \quad M_t(\beta) + v\beta = 0$$

respectively. The definite integral solution is now

$$u = \int_{v_1}^{v_2} \alpha_v(x) \beta_v(t) \phi(v) dv \quad \dots \quad (9)$$

where $\phi(v)$ is an arbitrary function and v_1 , v_2 suitable constants.

* Carslaw's *Fourier's Series and Integrals*, Ch. VIII. If the function $\chi(x)$ has a finite discontinuity at x , the integral in (6) represents

$$\frac{1}{2} \lim_{\epsilon \rightarrow 0} [\chi(x + \epsilon) + \chi(x - \epsilon)]$$

In the case of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots \quad (10)$$

a definite integral solution may be built up from the particular solution

$$u = f(x \cos \alpha + y \sin \alpha + iz) \quad .$$

where α is an arbitrary constant and f is an arbitrary function.

A more general solution is

$$u = \int_{\alpha_1}^{\alpha_2} f(x \cos \alpha + y \sin \alpha + iz, \alpha) d\alpha$$

where α_1, α_2 are constants. If we take $\alpha_1 = 0, \alpha_2 = 2\pi$, we obtain Whittaker's solution

$$u = \int_0^{2\pi} f(x \cos \alpha + y \sin \alpha + iz, \alpha) d\alpha$$

where $f(\xi, \eta)$ is an arbitrary function of ξ and η .

If in the preceding integral we integrate round a closed contour and make a special choice of the function f , we obtain

$$u = \frac{1}{2\pi i} \int_c \frac{\varphi(\alpha) d\alpha}{x \cos \alpha + y \sin \alpha + iz + \psi(\alpha)}$$

Let us suppose that the contour contains only one root of the equation

$$x \cos \alpha + y \sin \alpha + iz + \psi(\alpha) = 0 \quad \dots \quad (11)$$

then the integral may be evaluated by Cauchy's theorem, and we obtain Forsyth's solution

$$u = \frac{\varphi(\alpha)}{\psi'(\alpha) + y \cos \alpha - x \sin \alpha} \quad \dots \quad (12)$$

where α is a root of equation (11). This result may be compared with that of § 66.

If in Whittaker's solution we assume that $f(\xi, \eta)$ is independent of η and put $x = \rho \cos \varphi, y = \rho \sin \varphi$, we obtain

$$u = \int_0^{2\pi} f(\rho \cos \alpha - \varphi + iz) d\alpha$$

Writing $\alpha = \varphi + \omega$, the integral transforms into

$$u = \int_0^{2\pi} f(\rho \cos \omega + iz) d\omega \quad \dots \quad (13)$$

provided f is a single-valued function. This last integral represents a potential which is independent of φ or symmetrical about the axis of z . It is easily seen to be an integral of the second type, for $f(\rho \cos \omega + iz)$ is not a solution of Laplace's equation. It should be noticed that when $\rho \rightarrow 0$ this solution u generally reduces to $2\pi f(iz)$.

Another type of solution of Laplace's equation may be obtained as follows:

We have seen in § 66 that if $F_m(s)$ is a solution of Bessel's equation

$$\frac{d^2 F}{ds^2} + \frac{1}{s} \frac{dF}{ds} + \left(1 - \frac{m^2}{s^2}\right) F = 0$$

and k and m are arbitrary constants, then

$$u = e^{\pm kz + im\phi} F_m(k\rho)$$

is a solution of Laplace's equation. When $m = \frac{1}{2}$, we may take

$$F_m(k\rho) = (k\rho)^{-\frac{1}{2}} e^{ik\rho}$$

hence

$$u = e^{\pm k(z \pm i\rho)} \rho^{-\frac{1}{2}} e^{ik\phi}$$

is a solution for all values of k , and this indicates that

$$u = f(z \pm i\rho) \rho^{-\frac{1}{2}} e^{ik\phi} \quad \dots \quad (14)$$

is a solution when f is an arbitrary function. In particular, we obtain the solution

$$\begin{aligned} u &= \left(\frac{1}{z - i\rho} - \frac{1}{z + i\rho} \right) \rho^{-\frac{1}{2}} e^{ik\phi} \\ &= \frac{2i\rho^{\frac{1}{2}} e^{ik\phi}}{z^2 + \rho^2} = \frac{2i\sqrt{x+iy}}{x^2 + y^2 + z^2} \end{aligned}$$

Generalising this by a transformation of rectangular axes, we obtain the solution

$$u = \frac{[l(x - \xi) + m(y - \eta) + n(z - \zeta)]^{\frac{1}{2}}}{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \quad \dots \quad (15)$$

where ξ, η, ζ are arbitrary constants and l, m, n are constants such that $l^2 + m^2 + n^2 = 0$.

Now, let $\xi, \eta, \zeta, l, m, n$ be regarded as functions of a parameter θ , and let us integrate the expression

$$V = \frac{1}{2\pi i} \int u(x, y, z, \theta) F(\theta) d\theta \quad \dots \quad (16)$$

round a closed contour containing only one root of the equation

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0 \quad \dots \quad (17)$$

The integral may be evaluated by means of Cauchy's theorem, and we obtain the solution

$$V = -\frac{1}{2} F(\theta) \frac{[l(x - \xi) + m(y - \eta) + n(z - \zeta)]^{\frac{1}{2}}}{\xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta)}$$

where θ is a root of equation (17) and ξ', η', ζ' denote the derivatives of ξ, η, ζ with regard to θ .

In the particular case when l, m, n also satisfy the relation

$$l\xi' + m\eta' + n\zeta' = 0$$

the preceding solution may be expressed in a simpler form, for it is possible to find quantities λ, μ, ν depending only on θ and such that

$$[l(x - \xi) + m(y - \eta) + n(z - \zeta)][\lambda(x - \xi) + \mu(y - \eta) + \nu(z - \zeta)] \\ = [\xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta)]^2$$

It is sufficient, in fact, to write

$$\lambda = \frac{l(l\eta'\zeta' + m\zeta'\xi' + n\xi'\eta') - 2l^2\eta'\zeta'}{lmn}$$

and similar equations. It is easy to verify that

$$\lambda\xi' + \mu\eta' + \nu\zeta' = 0 \\ \lambda^2 + \mu^2 + \nu^2 = 0$$

Hence, if λ, μ, ν are functions of θ satisfying these equations, the function

$$V = \frac{F(\theta)}{[\lambda(x - \xi) + \mu(y - \eta) + \nu(z - \zeta)]^{\frac{1}{2}}} \quad \dots \quad (18)$$

is a solution of Laplace's equation. This is the result to which we have referred in § 66.

The general method of solving linear partial differential equations by integrals of the second type may be described briefly as follows.

Let $L_{x,y}(u) = 0 \quad \dots \quad (19)$

be the given linear partial differential equation in which we shall suppose for simplicity that there are only two independent variables. The method is, however, not limited to this case.

Let us suppose that a solution of the generalised harmonic equation

$$L_{x,y}(v) = M_t(v) \quad \dots \quad (20)$$

is known, and that this particular solution v is not simply a multiple of a function of x, y and a function of t .

Now consider the definite integral

$$u(x, y) = \int_{t_1}^{t_2} v(x, y, t) \varphi(t) dt \quad \dots \quad (21)$$

Assuming that the definite integral can be differentiated under the integral sign by the rule of Leibnitz, we find that

$$L_{x,y}(u) = \int_{t_1}^{t_2} L_{x,y}(v) \varphi(t) dt \\ = \int_{t_1}^{t_2} M_t(v) \varphi(t) dt$$

Now let $\varphi(t)$ be a solution of the differential equation adjoint to $M_t(v) = 0$, then $M_t(v) \varphi(t) dt$ is an exact differential dW ; and if the limits t_1, t_2 are chosen so that W vanishes at both limits, the definite integral (21) will be a solution of the partial differential

equation (19). To illustrate this method, let us consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{n}{x} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0$$

It is easy to verify that the function

$$v = f(x \cos t \pm y)$$

satisfies the partial differential equation

$$x^2 \left[\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial x^2} - \frac{n}{x} \frac{\partial v}{\partial x} \right] = \frac{\partial^2 v}{\partial t^2} + (n-1) \cot t \frac{\partial v}{\partial t}$$

Now $\sin^{n-1} t$ is an integrating factor of the expression on the left-hand side. Multiplying by this factor, the expression becomes equal to

$$\frac{\partial}{\partial t} \left[\sin^{n-1} t \frac{\partial v}{\partial t} \right]$$

The quantity within square brackets is zero for $t = 0$ and $t = \pi$, provided $n > 0$. Hence we obtain the solution

$$u = \int_0^\pi f(x \cos t \pm y) \sin^{n-1} t \, dt$$

This result is due to Poisson.

EXAMPLES.

1. Prove that Laplace's equation is satisfied by a definite integral of type

$$u = \int_0^\infty e^{-a\alpha} J_0(a\rho) \varphi(a) \, da$$

where $\varphi(a)$ is an arbitrary function and $J_0(x)$ is the Bessel function of zero order. Evaluate the integral when $\varphi(a) = 1$.

2. Is a solution of Laplace's equation completely determined if it is known to depend only on z and ρ , and its value for $\rho = 0$ is given for all values of z ?

3. Prove that the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{n}{x} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0$$

is satisfied by a definite integral of type

$$u = \int_0^\infty f(x \cosh \alpha \pm y) \sinh^{n-1} \alpha \, d\alpha$$

provided $f(z)$ is subject to certain limitations. (E. W. Hobson.)

4. Prove that the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial t^2}$$

is satisfied by

$$V = \int_0^{\frac{\pi}{2}} F[(t-y) - (\rho-y) \sec^2 \alpha] \, d\alpha$$

where $\rho^2 = x^2 + y^2$, provided

$$F(-\infty) = 0 \quad \text{and} \quad \tan \alpha F'[t - y - (\rho - y) \sec^2 \alpha] \rightarrow 0$$

as $\alpha \rightarrow \frac{\pi}{2}$, $F'(\xi)$ denoting the derivative of $F(\xi)$.

5. Prove that the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \lambda^2 V$$

is satisfied by the definite integrals

$$V = \int_0^{2\pi} e^{\lambda(x \cos \alpha + y \sin \alpha)} \varphi(\alpha) d\alpha$$

$$V = \int_0^\infty e^{-\frac{x^2+y^2}{4t} - \lambda^2 t} F\left(\frac{x+iy}{t}\right) \frac{dt}{t}$$

6. Prove that if F is an even function subject to certain limitations, the equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial}{\partial x} \left(x \frac{\partial V}{\partial x} \right)$$

is satisfied by $V = \int_0^{\frac{\pi}{2}} [F(x^2 \sin \theta + t) + F(x^2 \sin \theta - t)] d\theta$

(J. H. M. Wedderburn.)

7. Prove that the equation

$$(x-y) \frac{\partial^2 z}{\partial x \partial y} - \beta' \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 0$$

is satisfied by

$$z = \int_x^y (u-x)^{-\beta} (y-u)^{-\beta'} \varphi(u) du + (y-x)^{1-\beta-\beta'}$$

$$\times \int_x^y (u-x)^{\beta'-1} (y-u)^{\beta-1} \psi(u) du$$

where φ and ψ are arbitrary functions and $0 < \beta < 1$, $0 < \beta' < 1$.
(Appell.)

8. When $\beta + \beta' = 1$, the differential equation considered in the last example is satisfied by

$$z = \int_0^1 \varphi[x + (y-x)t] t^{-\beta} (1-t)^{\beta-1} dt$$

$$+ \int_0^1 \psi[x + (y-x)t] t^{-\beta} (1-t)^{\beta-1} \log [t(1-t)(y-x)] dt$$

§ 84. *The Green's function of a linear differential equation with assigned boundary conditions.* Let $L_x(u)$ denote the self-adjoint linear differential expression

$$L_x(u) = \frac{d}{dx} \left(p \frac{du}{dx} \right) + qu$$

where $q(x)$, $p(x)$ and $p'(x)$ are continuous functions of x in a finite

real interval $a \leq x \leq b$. If u and v are two functions whose first two derivatives are continuous in the interval (a, b) , we have the so-called *Green's formula* *

$$\int_a^b \{vL_x(u) - uL_x(v)\} dx = \left[p \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]_a^b. \quad (1)$$

Now let $\gamma(x, \xi)$ be a function of x and ξ , which has a continuous second derivative with regard to x at all points of the interval (a, b) with the exception of the point $x = \xi$. At this point the derivative $\frac{\partial \gamma}{\partial x}$ is discontinuous in such a way that

$$\lim_{\epsilon \rightarrow 0} \left[\left(\frac{\partial \gamma}{\partial x} \right)_{x=\xi+\epsilon} - \left(\frac{\partial \gamma}{\partial x} \right)_{x=\xi-\epsilon} \right] = -1$$

If γ is a solution of the differential equation $L_x(u) = 0$ for all points of the interval (a, b) other than $x = \xi$, it is called a *ground-solution* † (Grundlösung) for the interval (a, b) . A ground-solution may be expressed in terms of two linearly independent solutions $u_1(x)$, $u_2(x)$ of the equation $L_x(u) = 0$ by means of the formula

$$\gamma(x, \xi) = Au_1(x) + Bu_2(x) - \frac{1}{2} \frac{|x - \xi|}{x - \xi} \frac{u_1(x)u_2(\xi) - u_1(\xi)u_2(x)}{u_2(\xi)u_1'(\xi) - u_1(\xi)u_2'(\xi)}$$

where A and B are arbitrary constants. These constants may be chosen so that $\gamma(x, \xi)$ satisfies certain supplementary conditions. If these involve the values of γ and its derivatives at the end points of the interval in a manner which will be specified later, the function

$$g(x, \xi) = \frac{\gamma(x, \xi)}{p(\xi)}$$

will be called a *Green's function* for the differential equation and the given supplementary conditions.

Now let $f(x)$ be a function which is continuous in the interval $a \leq x \leq b$, and let $u(x)$ denote a solution of the differential equation $L_x(u) + f(x) = 0$, which satisfies supplementary conditions of the type just mentioned and has a continuous second derivative for the interval $(a \leq x \leq b)$. Also, let $v(x)$ be used to denote the Green's function $g(x, \xi)$ for the differential equation $L(u) = 0$ and the same supplementary conditions. Then Green's formula gives

$$\begin{aligned} \int_a^{\xi-\epsilon} \{vL_x(u) - uL_x(v)\} dx &= \left[p \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]_a^{\xi-\epsilon} \\ \int_{\xi+\epsilon}^b \{vL_x(u) - uL_x(v)\} dx &= \left[p \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]_{\xi+\epsilon}^b \end{aligned}$$

* It is really due to Abel, *Crelle's Journal*, Bd. 2 (1827).

† Hilbert, *Göttinger Nachrichten* (1904).

Adding these equations and making $\varepsilon \rightarrow 0$, we find on making use of the equations

$$\begin{aligned} L_x(u) &= -f(x) & L_x(v) &= 0 & v(x) &= g(x, \xi) \\ u(\xi) &= \int_a^b f(x)g(x, \xi) dx + \left[p \left(g \frac{du}{dx} - u \frac{\partial g}{\partial x} \right) \right]_a^b. \end{aligned} \quad (2)$$

Thus the value of $u(\xi)$ is expressed linearly in terms of the values of $g(x, \xi)$ and its derivative with regard to x . In some cases $u(\xi)$ can be expressed by means of the definite integral alone. For instance, if the supplementary conditions state that $u = 0$ when $x = a$ and when $x = b$, then we have also $g(a, \xi) = g(b, \xi) = 0$, and the second term on the right-hand side of (2) vanishes.

The same thing happens when the supplementary conditions state that

$$h \frac{du}{dx} + ku = 0 \quad \text{for } x = a \qquad H \frac{du}{dx} + Ku = 0 \quad \text{for } x = b$$

where h, k, H and K are constants.

If $p(x)$ is zero when $x = a$, a condition which states that u and $\frac{du}{dx}$ must be finite for $x = a$ will cause $p \left(g \frac{du}{dx} - u \frac{\partial g}{\partial x} \right)$ to be zero when $x = a$. A similar condition for $x = b$ or a condition such as $H \frac{du}{dx} + Ku = 0$ for $x = b$ will then make the right-hand side of (2) zero.

If in any of these cases a Green's function exists, we can conclude that a solution of $L_x(u) + f(x) = 0$, which satisfies the supplementary conditions and possesses a continuous second derivative, is given uniquely by the formula

$$u(\xi) = \int_a^b f(x) g(x, \xi) dx \quad \dots \quad (3)$$

for $(a < \xi < b)$.

It should be remarked, however, that a Green's function does not always exist. This may be seen by considering the simple case when the supplementary conditions take one of the forms just specified.

Let $\alpha(x)$ be a solution of $L_x(u) = 0$, which satisfies the condition associated with the end point $x = a$, and is such that α and $\frac{d\alpha}{dx}$ are continuous functions of x throughout the interval $a \leq x \leq b$. Also, let $\beta(x)$ be a solution of $L(u) = 0$, which satisfies the condition associated with the end point $x = b$, and is such that β and $\frac{d\beta}{dx}$ are continuous functions of x throughout the interval $a \leq x \leq b$. If

$\alpha(x)$ is not a constant multiple of $\beta(x)$, the arbitrary constant which occurs in the general expression for $\alpha(x)$ may be chosen so that

$$\alpha'(\xi) \beta(\xi) - \alpha(\xi) \beta'(\xi) = \frac{1}{p(\xi)} \quad \dots \quad (4)$$

for we have $\frac{d}{d\xi} [p(\xi) \{ \alpha'(\xi) \beta(\xi) - \alpha(\xi) \beta'(\xi) \}] = 0$

It can now be shown that the function

$$\begin{aligned} g(x, \xi) &= \alpha(x) \beta(\xi) \quad (x \leq \xi) \\ &= \alpha(\xi) \beta(x) \quad (x \geq \xi) \end{aligned} \quad \dots \quad (5)$$

is a Green's function of $L_x(u)$ for the given supplementary conditions.* The function evidently satisfies the supplementary conditions, and it is clear from (4) that the product $p(\xi) g(x, \xi)$ is a *ground-solution* of the differential equation $L_x(u) = 0$. Hence the function satisfies our definition of a Green's function.

This method of constructing a Green's function breaks down when $\alpha(x)$ is a constant multiple of $\beta(x)$, for then it is no longer possible to satisfy the relation (4). It should be noticed that in this case the equation $L(u) = 0$ possesses a solution $\alpha(x)$, which satisfies the supplementary conditions and is such that $\frac{d\alpha}{dx}$ is a continuous function of x throughout the interval $a \leq x \leq b$. Furthermore, if in this case the equation $L_x(u) + f(x) = 0$ possesses a solution satisfying the supplementary conditions and having a continuous derivative for all points of the interval $a \leq x \leq b$, this solution is not unique; for if k is an arbitrary constant, $u(x) + k\alpha(x)$ is a second solution with similar properties. It should be remarked that a solution with the properties just mentioned only exists when $f(x)$ satisfies a certain condition.

The above method of constructing Green's functions cannot be used in every case. For instance, if we take the equation $\frac{d^2u}{dx^2} = 0$ and the supplementary conditions

$$u(0) = -u(1) \quad u'(0) = -u'(1)$$

a solution $\alpha(x)$ which satisfies the first is given by $\alpha(x) = x - \frac{1}{2}$, and a solution $\beta(x)$ which satisfies the second is given by $\beta(x) = 1$. The function

$$\begin{aligned} g(x, \xi) &= x - \frac{1}{2} \quad (x \leq \xi) \\ &= \xi - \frac{1}{2} \quad (x \geq \xi) \end{aligned}$$

does not, however, satisfy the supplementary conditions, and so is not the correct Green's function.

In the case when a Green's function can be constructed by Kneser's method, it appears that the function is a *symmetric function* of x

* A. Kneser, *Math. Ann.*, Bd. 63.

and ξ (i.e. it is unaltered when x and ξ are interchanged), and it is the only Green's function corresponding to the supplementary conditions. Indeed, if there were two different Green's functions $g(x, \xi)$ and $G(x, \xi)$, the difference $g(x, \xi) - G(x, \xi)$ would have a continuous derivative throughout the range $a \leq x \leq b$; it would also be a solution of the differential equation $L_x(u) = 0$, and would satisfy both boundary conditions. Such a function would have to be a constant multiple of both $\alpha(x)$ and $\beta(x)$; consequently, $\alpha(x)$ would be a constant multiple of $\beta(x)$, which is contrary to hypothesis.

It should be noticed that the function $g(x, \xi)$ as defined by equations (5) can have a perfectly definite value when ξ lies outside the interval $a \leq \xi \leq b$, but now the integral

$$u(\xi) = \int_a^b f(x)g(x, \xi) dx$$

no longer satisfies the equation $L_\xi(u) + f(\xi) = 0$. Instead of this it satisfies the equation $L_\xi(u) = 0$, for it is a constant multiple of either $\alpha(\xi)$ or $\beta(\xi)$. This theorem is analogous to the following well-known property of a potential function,

$$V = \iiint \frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{\frac{3}{2}}}$$

At points within the region of integration

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

is equal to $-4\pi\rho(x, y, z)$, but at points outside the region of integration it is zero.

The above theory of Green's functions can easily be extended to partial differential equations of elliptic type. When there are two independent variables (x, y) , the Green's function $g(x, y; \xi, \eta)$ is a solution of the differential equation which becomes infinite at the point $x = \xi, y = \eta$, like the function $\log \sqrt{(x - \xi)^2 + (y - \eta)^2}$. It also satisfies a linear boundary condition like $g = 0$ at points of a closed curve.

When there are three independent variables (x, y, z) , the Green's function is a solution of the partial differential equation which becomes infinite at (ξ, η, ζ) , like the function $\frac{1}{r}$, where r is the distance from (x, y, z) to (ξ, η, ζ) .

The Green's function also satisfies a linear boundary condition such as $g = 0$ at points of a closed surface.

These developments are given in books on potential theory for the case of Laplace's equation. A brief account of the general theory is given in H. B. Heywood and M. Fréchet's *L'équation de Fredholm et ses applications à la Physique Mathématique*, Paris (1912), pp. 121-140.

EXAMPLES.

1. Prove that the Green's function for the expression

$$\frac{d^2u}{dx^2} + \lambda^2 u$$

and the supplementary conditions $u = 0$ when $x = 0$ and when $x = 1$ is

$$\begin{aligned} g(x, \xi) &= \frac{\sin \lambda x \cdot \sin \lambda(1 - \xi)}{\lambda \cdot \sin \lambda} \quad (x \leq \xi) \\ &= \frac{\sin \lambda \xi \cdot \sin \lambda(1 - x)}{\lambda \cdot \sin \lambda} \quad (x \geq \xi) \quad (\text{Hilbert.}) \end{aligned}$$

2. A Green's function for the expression
- $\frac{d^2u}{dx^2} + \lambda^2 u$
- and the condition that the function should have the period 1 is

$$g(x, \xi) = -\frac{1}{2\lambda} \left[\sin \lambda |x - \xi| + \cot \frac{\lambda}{2} \cos \lambda(x - \xi) \right]$$

3. The Green's function for the expression
- $L_x(u) \equiv \frac{d^2u}{dx^2} - \lambda^2 u$
- and the conditions that
- $u(x)$
- is finite for
- $x = +\infty$
- and
- $x = -\infty$
- is

$$g(x, \xi) = \frac{1}{2\lambda} e^{-\lambda|x-\xi|} \quad (\text{Hilbert.})$$

§ 85. *Riemann's method*.* Let $L_{x,y}(u)$ be used to denote the differential expression

$$\frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu \quad . \quad . \quad . \quad (1)$$

where a, b, c are functions of x and y with first derivatives which are continuous in a region R of the (x, y) plane. The *adjoint expression* $\bar{L}_{x,y}(v)$ is defined by the property that

$$vL_{x,y}(u) - u\bar{L}_{x,y}(v) \equiv \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \quad . \quad . \quad . \quad (2)$$

where M and N are certain quantities which can be expressed in terms of u, v and their first derivatives.

The expression for $\bar{L}_{x,y}(v)$ is suggested at once by the form of the expression adjoint to an ordinary linear differential expression, § 18, and it is easy to verify that if

$$\bar{L}_{x,y}(v) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv \quad . \quad . \quad (3)$$

* *Göttinger Abhandlungen*, Bd. 8 (1860); *Gesammelte Werke*, p. 145. See also Darboux, *Théorie des Surfaces*, t. 2.

then equation (2) is satisfied with

$$M = auv + \frac{1}{2} \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) - \frac{\partial \chi}{\partial y}$$

$$N = buv + \frac{1}{2} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + \frac{\partial \chi}{\partial x}$$

where χ is an arbitrary function. We shall find it convenient to put $\chi = 0$.

Now, if C is a closed curve in the (x, y) plane which lies entirely within the region R , and if u and v , together with their first derivatives, are supposed to be continuous in R , we have by Green's theorem,*

$$\begin{aligned} \int_c (M dy - N dx) &= \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\ &= \iint [v L_{x,y}(u) - u \bar{L}_{x,y}(v)] dx dy \end{aligned}$$

where the double integral extends over the area bounded by the curve C .

Let u and v be chosen so that the equations

$$L_{x,y}(u) = 0 \quad \bar{L}_{x,y}(v) = 0$$

are satisfied; then $\int_c (M dy - N dx) = 0$

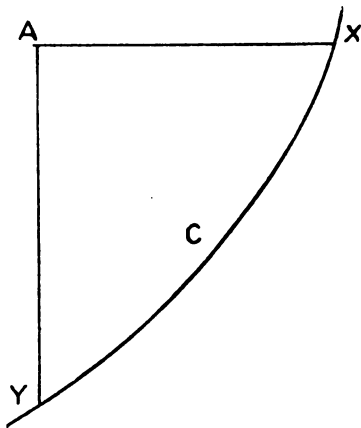


FIG. 8.

We shall now apply this theorem to the case in which the curve C consists of two lines AX , AY parallel to the axes of x and y respectively and a portion YX of a curve Γ .

* See Goursat-Hedrick, *Mathematical Analysis*, Vol. 1, p. 262.

We easily find that

$$\int_A^Y M dy + \int_Y^X (M dy - N dx) - \int_X^A N dx = 0$$

$$\begin{aligned} \text{Now } \int_A^Y M dy &= \int_A^Y \left[\frac{1}{2} \frac{\partial}{\partial y} (uv) - u \left(\frac{\partial v}{\partial y} - av \right) \right] dy \\ &= \frac{1}{2} [(uv)_Y - (uv)_A] - \int_A^Y u \left(\frac{\partial v}{\partial y} - av \right) dy \end{aligned}$$

Similarly,

$$\int_A^X N dx = \frac{1}{2} [(uv)_X - (uv)_A] - \int_A^X u \left(\frac{\partial v}{\partial x} - bv \right) dx$$

$$\begin{aligned} \text{Hence } (uv)_A &= \frac{1}{2} [(uv)_X + (uv)_Y] - \int_X^Y (M dy - N dx) \\ &\quad - \int_A^X u \left(\frac{\partial v}{\partial x} - bv \right) dx - \int_A^Y u \left(\frac{\partial v}{\partial y} - av \right) dy \end{aligned}$$

Now let us suppose that the function v can be chosen in such a way that

$$\frac{\partial v}{\partial x} - bv = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} - av = 0$$

on the lines AX, AY respectively ; then the value of u at the point A is given by the formula

$$(uv)_A = \frac{1}{2} [(uv)_X + (uv)_Y] + \int_Y^X (M dy - N dx)$$

It is easy to see from this equation that if the value of u and one of its derivatives is known for all points of the curve Γ , then the

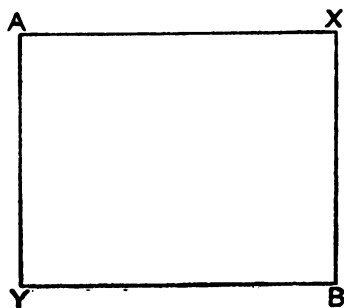


FIG. 9.

value of u at certain outside points can be found. Notice that if u and $\frac{\partial u}{\partial x}$ are given, the value of $\frac{\partial u}{\partial y}$ can be found from the equation

$$\frac{du}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds}$$

where s denotes the length of an arc of the curve, or, in fact, any parameter on which the coordinates of a point of the curve Γ depend. Everything depends of course on the possibility of finding a suitable function v . If (x_0, y_0) denote the coordinates of A , we may write

$$v = g(x, y; x_0, y_0)$$

and call it a *Green's function* of the differential expression $L_{x,y}(u)$.

Now let us consider the case when the curve Γ consists of a line BY parallel to the axis of x and a line BX parallel to the axis of y ; we then have

$$\begin{aligned} \int_Y^X (M dy - N dx) &= \int_B^X M dy - \int_Y^B N dx \\ \text{But } \int_Y^B N dx &= \int_Y^B \left[\frac{1}{2} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + buv \right] dx \\ &= \int_Y^B \left[-\frac{1}{2} \frac{\partial}{\partial x} (uv) + v \left(\frac{\partial u}{\partial x} + bu \right) \right] dx \\ &= \frac{1}{2} [(uv)_Y - (uv)_B] + \int_Y^B v \left(\frac{\partial u}{\partial x} + bu \right) dx \end{aligned}$$

Similarly,

$$\int_B^X M dy = \frac{1}{2} [(uv)_B - (uv)_X] + \int_B^X v \left(\frac{\partial u}{\partial y} + au \right) dy$$

Hence

$$(uv)_A = (uv)_B - \int_Y^B v \left(\frac{\partial u}{\partial x} + bu \right) dx - \int_B^X v \left(\frac{\partial u}{\partial y} + au \right) dy$$

Now let a function $u = h(x, y; x_1, y_1)$ be supposed to exist having the properties that

$$\frac{\partial u}{\partial x} + bu = 0 \quad \frac{\partial u}{\partial y} + au = 0$$

on the lines BY , BX respectively, and $L_{x,y}(u) = 0$ elsewhere.

If (x_1, y_1) are the coordinates of B , the arbitrary constant which occurs as a multiplier in the expression of the general function of the required type may be chosen so that $u_B = 1$, i.e.

$$h(x_1, y_1; x_1, y_1) = 1$$

We shall also suppose that the function g is chosen so that

$$g(x_0, y_0; x_0, y_0) = 1$$

Our formula then gives

$$h(x_0, y_0; x_1, y_1) = g(x_1, y_1; x_0, y_0)$$

Hence the Green's function $g(x, y; x_0, y_0)$ is a solution of the equation $\bar{L}_{x,y}(v) = 0$ when considered as a function of its first two arguments, and a solution of $L_{x_0,y_0}(u) = 0$ when considered

as a function of its last two arguments ; it is, in fact, a Green's function for each of the two adjoint partial differential equations.

To illustrate the method, let us consider the case when $a = 0$, $b = 0$; we then require a solution of the equation

$$\frac{\partial^2 v}{\partial x \partial y} + cv = 0$$

under the supplementary conditions $v = 1$ when $x = x_0$, $y = y_0$,

$$\frac{\partial v}{\partial x} = 0 \text{ when } y = y_0 \quad \frac{\partial v}{\partial y} = 0 \text{ when } x = x_0$$

Let us try $v = F[(x - x_0)(y - y_0)] = F(z)$

then
$$z \frac{d^2 F}{dz^2} + \frac{dF}{dz} + cF = 0$$

The supplementary conditions may be satisfied by assuming

$$F = 1 + \alpha_1 z + \alpha_2 \frac{z^2}{2!} + \dots + \alpha_n \frac{z^n}{n!} + \dots$$

We easily find that $n\alpha_n + c\alpha_{n-1} = 0$; hence

$$F = 1 - \frac{cz}{1^2} + \frac{c^2 z^2}{1^2 \cdot 2^2} - \dots = J_0(2\sqrt{cz})$$

where $J_0(\xi)$ is the Bessel function of zero order.

A solution of the equation $\frac{\partial^2 u}{\partial x \partial y} + cu = 0$ is now given in terms of the values, which it and one of its derivatives assume on a curve Γ , by means of the equation

$$2u(x_0, y_0) = u_x + u_y + \int_Y^X \left[\left(F \frac{\partial u}{\partial y} - u \frac{\partial F}{\partial y} \right) dy - \left(F \frac{\partial u}{\partial x} - u \frac{\partial F}{\partial x} \right) dx \right]$$

Certain equations which occur in mathematical physics may be reduced to the form $\frac{\partial^2 u}{\partial x \partial y} + cu = 0$ by simple substitutions, the telegraphic equation*

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t}$$

and the equation
$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} + a^2 c^2 w = 0$$

which occurs in the theory of the vertical propagation of waves in the atmosphere † may be mentioned as examples.

* For this equation see G. Kirchhoff, *Pogg. Ann.*, Bd. 100 (1857) ; O. Heaviside, *Phil. Mag.*, Aug. (1876) ; *Electrical Papers*, Vol. 1, p. 53 ; H. Poincaré, *Comptes Rendus*, t. 117 (1893), p. 1027 ; E. Picard, *Bull. de la Société math. de France*, t. 22 (1894), p. 2 ; M. I. Pupin, *Trans. Amer. Math. Soc.*, Vol. 1 (1900), p. 259.

† H. Lamb, *Proc. London Math. Soc.*, Ser. 2, Vol. 7 (1908) ; S. Sano, *Bull. of the Central Meteorological Observatory of Japan*, Vol. II. (1913), No. 2.

Riemann's method has recently been extended in such a way that it can be applied to linear partial differential equations of parabolic type.* Some of the results are given in the examples.

EXAMPLES.

1. The solution of $\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} + a^2 c^2 w = 0$

which satisfies the initial conditions

$$w = f(z) \quad \frac{\partial w}{\partial t} = \varphi(z) \quad \text{for } t = 0$$

is given by

$$w = \frac{1}{2c} \frac{\partial}{\partial t} \int_{s-\alpha}^{s+\alpha} J_0(as) f(\zeta) d\zeta + \frac{1}{2c} \int_{s-\alpha}^{s+\alpha} J_0(as) \varphi(\zeta) d\zeta$$

where a and c are real constants and $s^2 = c^2 t^2 - (z - \zeta)^2$.

(O. Heaviside and E. Picard.)

2. If w and $\frac{\partial w}{\partial t}$ are initially zero for $z > 0$, and the value of w at the plane $z = 0$ is maintained equal to $F(t)$, a solution of the partial differential equation considered in the last example is given by

$$\begin{aligned} w &= -c \frac{\partial}{\partial z} \int_0^{t-\frac{z}{c}} J_0(a\sigma) F(\tau) d\tau \quad t > \frac{z}{c} \\ &= 0 \quad t < \frac{z}{c} \end{aligned} \quad z > 0$$

where $\sigma^2 = c^2(t - \tau)^2 - z^2$. (S. Sano.)

3. If, on the other hand, the value of $\frac{\partial w}{\partial z}$ is maintained equal to $F(t)$ at $z = 0$, the corresponding value of w is

$$\begin{aligned} w &= -c \int_0^{t-\frac{z}{c}} J_0(a\sigma) F(\tau) d\tau \quad t > \frac{z}{c} \\ &= 0 \quad t < \frac{z}{c} \end{aligned}$$

4. Prove that the Green's function for the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta'}{x-y} \frac{\partial u}{\partial x} + \frac{\beta}{x-y} \frac{\partial u}{\partial y} = 0$$

is $g(x, y; x_0, y_0) = (y_0 - x)^{-\beta'} (y - x)^{\beta+\beta'} (y - x_0)^{-\beta} F(\beta, \beta', 1; \sigma)$

* Wera Lebedeff, *Dissertation Göttingen* (1906); E. E. Levi, *Annali di Matematica* (3), t. 14 (1908), p. 187; E. Holmgren, *Arkiv för Matematik*, Bd. III and IV (1907-8); E. Goursat, *Traité d'Analyse*, t. III; M. Gevrey, *Liouville's Journal* (6), t. 9 (1913), pp. 305-475, t. 10 (1914), pp. 105-148; G. C. Evans, *American Journal*, Vol. 37 (1915), p. 431.

where
$$\sigma = \frac{(x - x_0)(y - y_0)}{(x - y_0)(y - x_0)}$$

and F denotes the hypergeometric function. (Darboux.)

5. Prove that equation (2) of § 85 is satisfied if

$$\begin{aligned} L(u) &= \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} & \bar{L}(v) &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} \\ M &= v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} & N &= -uv \end{aligned}$$

6. Let C denote the curve (Fig. 10) formed of a segment $A_1 A_2$ of a line $y = \text{constant}$ and two curves C_1, C_2 through the points A_1, A_2 . Let

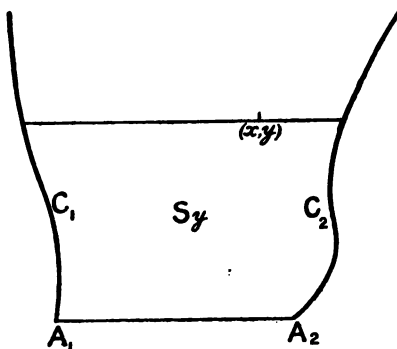


FIG. 10.

S_y denote the region bounded by C and a straight line through (x, y) parallel to the axis of x , also let C_y denote the boundary of this region.

Let z be a solution of the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = f(x, y)$$

such that the derivatives entering into this equation are continuous in S_y ; then, if

$$h(x, y; \xi, \eta) \equiv (y - \eta)^{-\frac{1}{2}} \exp \{ (x - \xi)^2 / 4(y - \eta) \}$$

we have

$$\int_{C_y} h \left\{ z(\xi, \eta) d\xi + \left[\frac{\partial z}{\partial \xi} - z \frac{x - \xi}{2(y - \eta)} \right] d\eta \right\} - \iint_{S_y} h \cdot f(\xi, \eta) d\xi d\eta = \lambda z(x, y)$$

where λ is $2\sqrt{\pi}$, $\sqrt{\pi}$, or 0 according as the point (x, y) lies inside, on the boundary, or outside the region bounded by C . If $x = X_1(y)$, $x = X_2(y)$ are the equations of the curves C_1 and C_2 the functions X_1 and X_2 are assumed to be continuous with their first derivatives and to have only a finite number of maxima and minima for the considered range of values of y . (This theorem has been established under less stringent conditions. See the memoirs cited above.)

CHAPTER XI

THE MECHANICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

§ 86. *The reduction of a differential equation to a convenient form.* Simple mechanisms for solving differential equations are generally suitable only for differential equations of certain special types, and so it is important to know in the first place whether the equations which can be reduced to these special types are of frequent occurrence.

One type of equation of the first order considered by E. Pascal* is

$$p = \Phi\{Q(x) - y\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Q and Φ are arbitrary functions; it is of special importance because C. Ajello† has shown that a general Riccattian equation

$$p = Ay^2 + By + C$$

can be reduced to the above form. The appropriate transformation is

$$\xi = - \int A\beta \, dx \quad \eta = \int C\gamma \, dx - y\gamma = Q(\xi) - y\gamma$$

where $\log \beta = \int B \, dx \quad \log \gamma = - \int B \, dx$

The reduced equation is

$$\frac{d\eta}{d\xi} = \{Q(\xi) - \eta\}^2$$

The first type of equation considered by Lord Kelvin ‡ is

$$\frac{d}{dx} \left\{ \frac{1}{U} \frac{dy}{dx} \right\} + y = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where U is an arbitrary function of x . The general linear differential equation of the second order

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0$$

* *Rend. R. Acc. delle Sc. fis. e mat. di Napoli* (3), v. 17 (1911); *Ibid.* (3), v. 18 (1912).

† *Ibid.* (3), v. 18 (1912).

‡ *Proc. Roy. Soc.*, Vol. 24 (1876).

may be reduced to this form by writing

$$dz = Qe^{\int P dx} dx \quad U(z) \cdot Qe^{2 \int P dx} = 1$$

Equation (2) may be solved by a method of successive approximations by writing

$$y_2 = \int_0^x U(C - \int_0^x y_1 dx) dx$$

$$y_3 = \int_0^x U(C - \int_0^x y_2 dx) dx$$

.

where y_1 is any function of x , to begin with, as, for example, $u_1 = x$; then y_2, y_3 , etc., are successive approximations converging to that one of the solutions of (2) which vanishes when $x = 0$.

To carry out the approximations mechanically an instrument is needed for the calculation of an integral of type

$$\int_0^x \varphi(x) \psi(x) dx$$

As Lord Kelvin pointed out, the integrating machine invented by James Thomson is quite suitable. Pascal has also constructed a suitable integrator by generalising that invented by Abdank-Abakanowicz.

EXAMPLES.

1. Reduce the differential equation

$$y' = y^n + P(x)$$

to Pascal's form with $\Phi(t) = t^n$.

2. Show that Abel's equation

$$y' = Ay^3 + By^2 + Cy + D$$

in which A, B, C, D are arbitrary functions of x , can be reduced to Pascal's form with $\Phi(t) = t^n$.

§ 87. Mechanisms for the solution of equations of Pascal's type.*

A rigid rectangle ABCD, made of brass, is partly supported by an axle with two equal wheels W_1 and W_2 , which enable the rectangle to move in a direction parallel to the axis of x . It is also supported by two very small wheels with sharp edges. These wheels are attached to two carriages H and G, which can slide along the rods AD, BC respectively, and the connections are made in such a way that the

* A full account is given in Pascal's book, *I miei integrali per equazioni differenziali*, Naples (1914). See also *Atti della R. Accademia delle scienze fisiche e matematiche di Napoli* (S. 2), v. 15 (1913), and the article by C. Tweedie in *Modern Instruments of Calculation*, G. Bell & Sons, London (1914).

planes of the small wheels are free to rotate round axes perpendicular to the plane of the paper.

In the first form of the apparatus a curved bar EGH is pivoted to the carriage G so that it is free to rotate, and is provided with a slot so that it can slide over a pivot in the carriage H. It is also connected with the small wheel at H in such a way that the plane of the wheel always touches the curve EHG at H, and is of course perpendicular to the plane of the paper.

As the wheel at G traces out the curve $y = Q(x)$, the rectangle moves parallel to the axis of x , while the carriage H moves along the bar AD and the curved bar EHG slides over the pivot at the

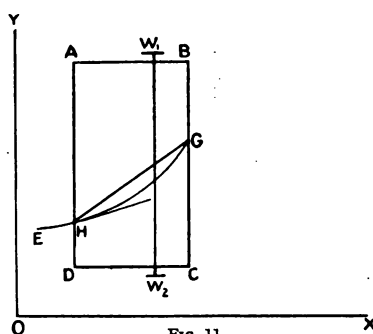


FIG. 11.

same time changing the orientation of the small wheel at H so that its plane always contains the tangent at H.

Let y denote the ordinate of H and β the angle between HG and the tangent at H. If HG makes an angle α with Ox, we have

$$y' = \tan(\alpha - \beta) \quad a \tan \alpha = Q(x) - y \quad (1)$$

where $AB = a$. Now, since the curve EHG retains its shape, the angle β depends only on the difference of the ordinates of G and H; consequently we may write

$$\tan \beta = F\{Q(x) - y\}. \quad (2)$$

Hence it follows that H traces out a curve whose ordinate y satisfies an equation of Pascal's type with

$$\Phi(t) = \frac{t - aF(t)}{a + tF(t)}$$

To determine the shape of the curve EHG when $\Phi(t)$ is given, we take G as pole and use polar coordinates ρ, θ ; then

$$\frac{\rho}{d\rho} \frac{d\theta}{d\rho} = F(\sqrt{\rho^2 - a^2})$$

The polar equation of the curve is thus found by direct integration. By adjusting the arbitrary constant we can make one solution of the differential equation satisfy certain supplementary conditions.

In the second form of the apparatus the curved bar is rigidly connected to the small wheel attached to the carriage H, and is slotted so that it slides over a pivot in the carriage G (Fig. 12).

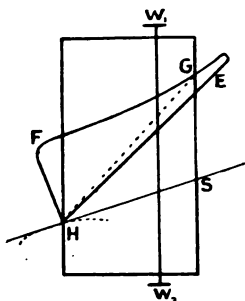


FIG. 12.

In this case the tangent to the path of H is again in the direction of the wheel at H, and makes a fixed angle, say 90° , with the portion FH of the curved bar.

We again have the equations (1) and (2), but now if $HG = \rho$, the polar equation of the locus of G is simply

$$\tan \beta = F(\sqrt{\rho^2 - a^2}) \quad \theta = \beta$$

H being the pole.

§ 88. *Mechanisms for integrating the product of two functions.*

In the integrating machine of Abdank-Abakanowicz two carriages H and G slide along the sides of a rectangle ABCD as before. The

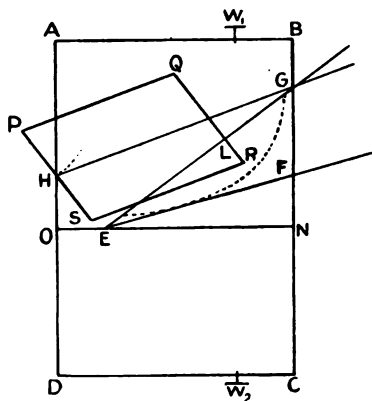


FIG. 13.

small wheel at H is, however, fixed with its plane at right angles to one arm PHS of an articulated parallelogram PQRS, consequently the tangent to the path of H is perpendicular to PH and parallel

to a bar ELG which passes through a cylindrical tube at L so as to maintain QR at right angles to EL .

This bar ELG slides over a pivot in the carriage G , and turns round a fixed point E on the axis of x at distance a from N . As the wheel of the carriage G traces out the curve $y = \varphi(x)$, the rectangle moves parallel to the axis of x and the wheel of the carriage H traces a curve for which $\frac{dy}{dx} = \frac{GN}{EN} = \frac{\varphi(x)}{a}$. The bar HG turns about a pivot attached to the carriage H and slides over a pivot in the carriage G . The curve traced out by the wheel at H is the integral curve of the function $\frac{\varphi(x)}{a}$.

Pascal has modified this apparatus by replacing the fixed pivot E by one which can move along a slot in the bar ON , the motion of E being governed by a bar EF which slides over a pivot in the movable carriage F , which slides along BC in such a way that F traces out the curve $y = \psi(x)$ and EF always touches the rectangular hyperbola defined by the property $EN \cdot NF = 1$. We then have $a\psi(x) = 1$, and so the path of H is the integral curve of the function $\varphi(x)\psi(x)$.

The integrating machine invented by James Thomson is based on an entirely different principle. Motion is transmitted from a disk to a cylinder by the intervention of a loose ball, which presses by its gravity on the disk and cylinder, the pressure being sufficient to give the necessary frictional coherence at each point of rolling contact; and the axis of the disk and that of the cylinder being both held fixed in position by bearings in stationary framework, and the arrangement of these axes being such that when the disk and the cylinder are kept steady, or, in other words, without rotation on their axes, the ball can roll along them in contact with both, so that the point of rolling contact between the ball and the cylinder shall describe a straight line on the cylindric surface parallel necessarily to the axis of the cylinder—and so that the point of rolling contact of the ball with the disk shall traverse a straight line passing through the centre of the disk. It will thus readily be seen that, whether the cylinder and the disk be at rest or revolving on their axes, the two lines of rolling contact of the ball, one on the cylindric surface and the other on the disk, when both considered as lines traced out in space fixed relatively to the framing of the whole instrument, will be two parallel straight lines, and that the line of motion of the ball's centre will be straight and parallel to them. For facilitating explanations, the motion of the centre of the ball along its path parallel to the axis of the cylinder may be called the ball's longitudinal motion.

Now for the integration of $y dx$: the distance of the point of contact of the ball with the disk from the centre of the disk in the ball's longitudinal motion is to represent y , while the angular space

turned by the disk from any initial position represents x ; and then the angular space turned by the cylinder will, when multiplied by a suitable constant numerical coefficient, express the integral in terms of any required unit for its evaluation. The plane of the disk may suitably be placed inclined to the horizontal at some such angle as 45° .

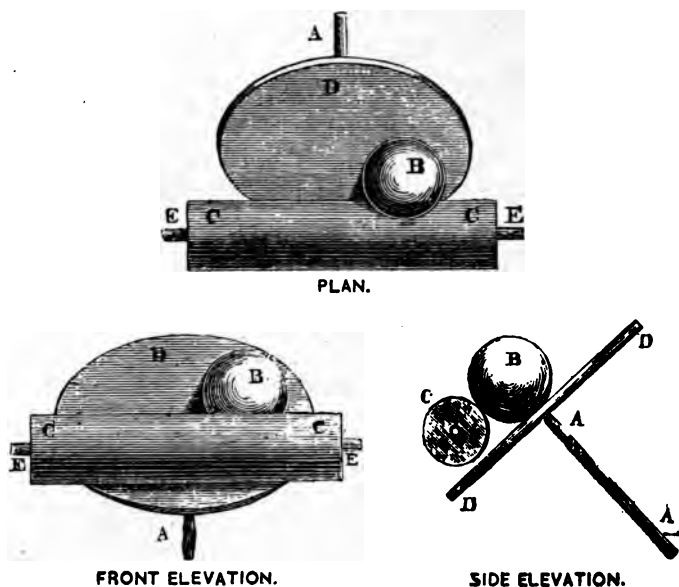


FIG. 14.

To calculate $\int \varphi(x)\psi(x) dx$, the rotating disk is to be displaced from a zero or initial position through an angle equal to $\int_0^x \varphi(x) dx$, while the rolling globe is moved so as always to be at a distance from its zero position equal to $\psi(x)$. This being done, the cylinder turns through an angle equal to $\int_0^x \varphi(x)\psi(x) dx$.

One way of giving the required motions to the rotating disk and rolling globe is as follows :

On two pieces of paper draw the curves

$$y = \int_0^x \varphi(x) dx \quad \text{and} \quad y = \psi(x)$$

Attach these pieces of paper to the circumference of two circular cylinders, or to different parts of the circumference of one cylinder, with the axis of x in each in the direction perpendicular to the axis of the cylinder. Let the two cylinders (if there are two) be geared

Two operators are required as one operator could not move the two points so as to fulfil this condition—at all events unless the motion were very slow. One of these points, by proper mechanism, gives an angular motion to the rotating disk equal to its own linear motion, the other gives a linear motion equal to its own to the centre of the rolling globe.

§ 89. *Pascal's mechanism for the solution of a linear equation of the first order.*

FIG. 15.

$$\mathbf{E} \mathbf{N} \cdot \mathbf{E} \mathbf{N} = 1$$

If now as the rectangle moves the points G, M, F trace out the curves $y = f(x)$, $y = \phi(x)$, $y = -F(x)$ respectively, the point H

will describe the graph of the function $\int_0^x f(x)F(x) dx$, and since $HK = GM$, we have

$$f(x) = \int_0^x f(x)F(x) dx + \varphi(x)$$

with $f(0) = \varphi(0)$. Differentiating, we see that $y = f(x)$ is a solution of the differential equation

$$\frac{dy}{dx} = yF(x) + \varphi'(x)$$

§ 90. *The solution of a linear differential equation of the second order.** Let the differential equation be

$$z'' = \Phi(x)z' + \varphi'(x)z + \psi(x)$$

where $\Phi(x)$, $\varphi'(x)$ and $\psi(x)$ are assigned functions of x representable by curves. The equation may be replaced by the two linear equations

$$y' = F(x)y + \psi(x) \quad z' = f(x)z + y \quad \dots (1)$$

where

$$f + F = \Phi \quad f' = Ff + \varphi' \quad \dots (2)$$

To solve the last two equations we use a simple modification of the last apparatus in which pivots at K and a point R on AD govern the motions of those at F and G with the aid of two slotted bars which pass over RF and KG respectively and also over a pivot O which slides along a slot in a rod equidistant from AD and BC . This makes $KR = FG$, and so if $KR = \Phi(x)$ the points G and F will trace the graphs of functions $f(x)$ and $F(x)$ respectively which satisfy the equations (2). The two equations (1) may then be integrated by the preceding method.

It should be noticed that

$$f' + f^2 = \Phi f + \varphi'$$

consequently the function $f(x)$ satisfies a Riccattian equation, and so we have another mechanism for the solution of an equation of this type. A third mechanism is described by Tweedie (*loc. cit.*).

A device for solving certain types of differential equations with the help of a liquid has been described by Petrovitch.†

EXAMPLES.

1. Devise an apparatus with a curved slotted bar for the solution of a differential equation of type

$$ay' = F\{x + \varphi(y)\}$$

where a is a constant and F and φ arbitrary functions. (E. Pascal.)

* E. Pascal, *Giornale di Matematiche*, Vol. LIV (1916), p. 132.

† *Amer. Journ. of Math.* Vol. 20, p. 291 (1898); Vol. 22, p. 1. The differential equations solved by means of Petrovitch's apparatus are of the same types as those solved by Pascal's mechanisms.

2. Devise an apparatus for the solution of an equation of type

$$\frac{dy}{dx} = \frac{y^2 - 1}{f(x) - y} \quad (\text{E. Pascal.})$$

3. Reduce the differential equation

$$\frac{dv}{dx} = \frac{v[\sin \alpha + \psi(v)]}{\cos \alpha}$$

to the form given in the last equation. (Hayashi.)

4. In Fig. 13 an additional curved bar HU normal to PS is provided with a slot and slides over a pivot U which is attached to a carriage that can move up and down BC. The carriage F is maintained at a constant distance from U, and E moves so that $EN \cdot NF = 1$ as before. Prove that this mechanism can be used to solve a differential equation of the form

$$y = p\chi(x) + f(p)$$

where $p = \frac{dy}{dx}$ and the functions χ and f are arbitrary.

5. In Fig. 13 a straight slotted bar passes over movable pivots at H, E and K, the last named being attached to a carriage which moves up and down BC, and is made to move along a given curve. Prove that this apparatus can be used to solve a linear differential equation of the first order.

6. A slotted bar OPQR turns round a fixed point O and slides over a pivot P fixed on the side AB of a rigid rectangle ABCD, which is supported on wheels so that it can move only in a direction at right angles to AB. A rolling wheel W with a sharp edge is attached to a carriage Q, which can run along the slot in OR. The wheel W is compelled by means of an articulated parallelogram MNHK to turn about a horizontal axis parallel to a second slotted bar ULHK, which turns round a pivot at C and passes over a second pivot L fixed to a carriage which can run along a slot in AB. The side HK of the articulated parallelogram is attached to a carriage which can run along a slot in the bar UL. The rolling wheel attached to the carriage L is made to describe a given curve. Prove that this mechanism can be used to solve a differential equation of type

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

(W. von Dyck, *Abhandlungen d. K. Bayerischen Akad. d. Wiss.* Bd. 26, 1914.)

7. A slotted bar AB is supported on wheels so that it can move only in a direction at right angles to itself. A carriage C runs along the slot and is provided with a rolling wheel W, whose plane makes a constant angle with a second bar CP of constant length. A rolling wheel at P describes a given curve. Find the differential equation of the curve described by C. (A. Scribanti.)

MISCELLANEOUS EXAMPLES.

1. Solve the equations
- $$\frac{dy}{dx} = \frac{y(1-y^2)}{x(1-x^2)}$$
- $$\frac{dy}{dx} = \frac{y(1+x)}{x(1+y)}$$
- $$\frac{dx}{dt} = k \frac{a-x}{x}$$
- $$x \dot{dy} + y^2 dx = 0$$

2. The population of a country at time t is P , the number of individuals affected at this time by a certain disease is Z , a proportion $h \cdot dt$ of the non-affected become affected in the next short interval of time dt , and a proportion $r dt$ of the affected become unaffected, *i.e.* revert to the non-affected group.

If $n dt$, $m dt$, $i dt$, $e dt$ denote respectively the nativity, mortality, immigration, and emigration rates of the non-affected part of the population in time dt , and $N dt$, $M dt$, $I dt$, $E dt$ denote the similar rates among the affected parts, prove that

$$\frac{dP}{dt} = vP - (v - V)xP$$

$$\frac{d}{dt}(xP) = hP(1-x) + (V - N - r)xP$$

where $v = n - m + i - e$ $V = N - M + I - E$ $Z = xP$.

Obtain a differential equation for x and solve it (1) on the supposition that h , v , V , N , r are constants; (2) on the supposition that $h = cx$, where c is a constant. (Sir Ronald Ross.)

3. The equation for a reversible bimolecular chemical reaction is

$$\frac{dx}{dt} = k_1(a-x)^2 - k_2x^2$$

where k_1 , k_2 and a are constants. Solve this equation with the aid of the substitution

$$y = \frac{x}{a-x}$$

4. Solve the pairs of equations

$$\frac{dy}{dt} = k_1(a-y) \quad \frac{dz}{dt} = k_2(y-z)$$

$$\frac{dx}{dt} = k_1(a-x)(b-x-y) \quad \frac{dy}{dt} = k_2(x-y)(b-x-y)$$

and suggest types of consecutive chemical reactions which will lead to these equations.

5. Solve the equation

$$\frac{dy}{dx} - y \left(\frac{4}{x} - \frac{2x}{x^2 + a^2} \right) = - \frac{2ax}{x^2 + a^2}$$

and prove that the particular integral which vanishes when $x = a$ is given by

$$y = \frac{x^2}{a} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)$$

6. Find the law according to which a man's daily income must increase in order that he may allow a fixed amount for the daily rate of increase of the cost of living and increase his daily savings by a fixed percentage of his daily income. (Treat the length of a day and the daily variations as infinitesimal.)

7. A scout running at a constant rate u endeavours to keep an obstacle between himself and a sentry who is walking along a straight line with velocity v . Find a differential equation for the path of the scout.

8. Solve the equation $\frac{dy}{dx} - \frac{y}{x} = - \frac{x^3}{a^2 y}$

where a is a constant, and determine the arbitrary constant so that $y = 0$ when $x = b$.

9. Solve the equations $\frac{d^2 y}{dt^2} = \frac{dx}{dt} - a$

$$\frac{d^2 x}{dt^2} = - \frac{dy}{dt}$$

where a denotes a given constant, and determine the arbitrary constants so that for $t = 0$, we have

$$x = 0 \quad y = 0 \quad \frac{dx}{dt} = 2a \quad \frac{dy}{dt} = 0$$

10. Find the general solution of the pair of equations

$$\frac{dx}{dt} + 16y - 6x = 48 \cos 2t$$

$$\frac{dy}{dt} + 2y - x = 6 \cos 2t - 2 \sin 2t$$

11. Solve the equations

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} - 2y = 2 \sin x + \cos x - 2x - 3$$

$$\frac{d^4 y}{dx^4} + 3 \frac{d^2 y}{dx^2} - 4y = (1 + x) \cos 2x + e^x$$

12. If $x = 0$, $y = 0$ for $t = 0$ and

$$\frac{dx}{dt} = x - y - t - 2 \quad \frac{dy}{dt} = y + t$$

express x and y as functions of t .

13. Solve the equation

$$\frac{d^3y}{dx^3} - 3 \frac{d^2y}{dx^2} + \frac{dy}{dx} - 3y = \sin x$$

and determine the integral curve so that it may have a point of inflexion at the origin, the flex tangent being the axis of x .

14. A floating body oscillates in a vertical direction about a position of equilibrium. Prove that its equation of motion is $V''z + gAz = 0$, where A is the area of the plane of floatation and V the volume of the body below this plane in the position of equilibrium. It is assumed that the vertical line joining the centre of gravity of the body to the centre of buoyancy (centre of gravity of fluid displaced) passes through the centre of gravity of the plane of floatation.

15. Water flows into a cistern at a constant rate and flows out at the bottom with a velocity whose square is proportional to the height of the water in the cistern. Find how much water has flowed out of the cistern at the end of an interval of time t , assuming that the cistern was initially empty.

16. Solve the equations

$$\frac{dx}{dt} = y + z \quad \frac{dy}{dt} = z + x \quad \frac{dz}{dt} = x + y$$

17. A grocer begins to sell a box of N storage eggs which go bad at a rate proportional to the number present. Initially customers buy the eggs at rate c , but as the grocer only destroys a percentage k of the bad eggs, and the customers get the rest, the rate of purchase varies at a rate which is equal to a times the rate at which good eggs are bought minus b times the rate at which bad eggs are bought. Find the number of eggs which the grocer has left at the end of an interval t .

18. Determine the constant
- λ
- so that the expression

$$\frac{x^2}{x + \lambda y} dx + \frac{3y^2}{y + \lambda x} dy$$

may be the exact differential of a function $U(x, y)$, and determine this function.

Find the orthogonal trajectories of the curves $U(x, y) = \text{constant}$.

19. If $u = y^2 - x$, find the condition that $f(u)$ may be an integrating factor of $P dx + Q dy$. Consider the case when

$$P = (2y^2 - 2x - 1)e^x + (2y^2 - 3x)e^y$$

$$Q = 2ye^x + 2x(y^2 + y - x)e^y$$

20. Solve the equations

$$2xy \frac{dy}{dx} = k(x^2 + y^2)$$

$$2x^3 \frac{dy}{dx} = y(3x^2 + y^2)$$

In each case form and integrate the equation of the orthogonal trajectories.

21. Integrate the equation

$$(x + 2x^\alpha + y^\alpha) dy + y(1 - x) dx$$

knowing that it admits an integrating factor of type P^α , where α is a constant and P a polynomial of the first degree in x .

22. Solve the equation

$$(x^2 - y^4) \frac{dy}{dx} - xy = 0$$

by taking y and $\frac{x}{y}$ as new variables.

23. Solve the equations

$$\frac{y}{p} - x = \alpha y^2$$

$$2y = x \left(p + \frac{1}{p} \right) + \frac{1}{3} p^3 + 1 - p$$

$$y - 2xy' + y^2 y'^2 = 0$$

by differentiation.

24. The orthogonal trajectories of a family of similar and similarly situated curves having O as centre of similitude form another family of similar and similarly situated curves with O as centre of similitude.

25. Solve the equation

$$xyp^2 + (x^2 - y^2 - c^2)p - xy = 0$$

by putting

$$x = u \cos t - \frac{du}{dt} \sin t \quad y = u \sin t + \frac{du}{dt} \cos t$$

26. A particle oscillates in a straight line about a centre of force, the force varying as the distance, and is subject to a retardation $kx(\text{vel})^2$. Write down the equation of motion, and show that if a and b are two successive elongations, on opposite sides of the centre, then

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}$$

(Lamb's *Dynamics*, p. 299, Ex. 14.)

27. Solve the equation

$$x^2 y'' - (2\beta - 1)xy' + (\beta^2 + 1)y = x \log x \cdot \cos(\log x)$$

28. Determine λ so that the equation

$$x^3 \frac{d^2 y}{dx^2} = \lambda y$$

may be satisfied by $y = \sqrt{x}$, and then solve the equation.

29. Find the condition that if $y = u(x)$ is a solution of the equation

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0$$

then

$$y = \frac{du}{dx} \text{ is also a solution}$$

30. Solve the equation

$$y = xp^2 + \frac{2p-1}{(p-1)^2}$$

by differentiating with regard to x and taking p as the new independent variable.

31. Find an integrating factor of the equation

$$(a^2 - yz) \frac{dx}{x} + (a^2 + xz) \frac{dy}{y} + (y - x) dz = 0$$

32. Determine the function ϕ so that the expression

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \phi(x, y, z) dz$$

may be an exact differential. Show also that any total differential expression may be reduced to the above form by multiplying it by a suitable factor.

33. Solve the equations

$$2y(2a - x)p + (x^2 + z^2 - y^2 - 4ax)q + 2yz = 0$$

$$xp + yq = \frac{2\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2} - a}$$

and prove that in the latter case the integral surfaces are ruled surfaces.

34. Solve the equation

$$\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

where

$$u = x(x + y + z) \quad v = y(x + y + z) \quad w = z(x + y + z)$$

and verify that if points move according to the law

$$\frac{dx}{dt} = u \quad \frac{dy}{dt} = v \quad \frac{dz}{dt} = w$$

then the points which are on one integral surface of the partial differential equation at time $t = 0$ will be on another integral surface of the partial differential equation at time t .

35. Solve the partial differential equations

$$xp + 3yq = \frac{4y^2}{x^2}$$

$$p^2 + q^2 = \frac{3a^2}{z^2}$$

$$p = q + \frac{1}{2}q^2$$

36. Determine a set of solutions of the equations

$$\frac{dx'}{2x'y'} = \frac{dy'}{y'^2 - z'^2 - x'^2} = \frac{dz'}{2y'z'} = dt$$

so that

$$x' = x \quad y' = y \quad z' = z \quad \text{for } t = 0$$

Show that the transformation $x' = f(x, y, z, t)$, $y' = g(x, y, z, t)$, $h' = h(x, y, z, t)$ makes the equation $dx'^2 + dy'^2 + dz'^2 = 0$ a consequence of $dx^2 + dy^2 + dz^2 = 0$, and interpret this result geometrically.

37. Prove that the function $V = \log(r + z)$ satisfies Laplace's equation $\nabla^2 V = 0$, and verify that the fundamental solution $\frac{1}{r}$ may be derived from it by differentiation with regard to z .

38. Prove that Maxwell's electromagnetic equations may be satisfied by writing

$$\alpha + iX = \frac{\partial(\sigma, \tau)}{\partial(y, z)} \quad \beta + iY = \frac{\partial(\sigma, \tau)}{\partial(z, x)} \quad \gamma + iZ = \frac{\partial(\sigma, \tau)}{\partial(x, y)}$$

$$\text{where} \quad \sigma = \log \frac{x + iy}{r - z} \quad \tau = t - \frac{r}{c}$$

Show also that the electric and magnetic forces in the field thus specified are at right angles to the radius from the origin, and are also at right angles to one another.

(This electromagnetic field is identical with that considered by O. Heaviside, *Electromagnetic Theory*, vol. iii. p. 122. There is a continual production at the origin of oppositely charged particles of electricity which fly away with the velocity of light, travelling along the axis of z in opposite directions. The electric and magnetic forces at any point are both inversely proportional to the distance from the axis of z . The flow of energy* is rectilinear and directed away from the origin so that the field has the character of a radiant field.)

39. Prove that the electrostatic field of a point charge can be derived from a radiant field of the above description by differentiation with regard to z . Hence show that two superposed radiant fields may in the limit be equivalent to the electrostatic field of a point charge.

40. Prove that a solution of Maxwell's equations may be obtained by multiplying the expressions for $\alpha + iX$, $\beta + iY$, $\gamma + iZ$ in Ex. 38 by $f(\sigma, \tau)$, where f is an arbitrary function.

41. Generalise the result of Ex. 38 so as to obtain a specification of a radiant field in which there is a continual production of electrified light-particles† at a moving point S . Can the field of a moving point charge of electricity be built up from two superposed radiant fields?

42. Light-particles move to and from a moving point S in such a way that if S_1, S_2, S_3 are any three consecutive positions of the moving point, the light-particle which arrives at S_3 encounters on its journey the light-particles which started from S_1 and S_2 . Obtain a set of three differential equations for the variation of the direction cosines of the line along which a light-particle moves when it leaves S , and prove that the light-particles which leave S lie at any instant on a line of electric force of the field of a point charge of electricity carried by S .

* This is in a direction at right angles to both the electric and magnetic forces.

† A light-particle will be understood to mean a particle which travels along a straight line with the velocity of light.

43. The temperature at the surface of the ocean is supposed to be represented by the real part of the function $F(x, y)e^{i\gamma t}$, the surface being regarded as the plane $z = 0$. Experiments indicate that the temperature oscillation lags more and more in phase as the depth below the surface increases. Assuming that the temperature of the ocean at a depth z below the surface is governed by a partial differential equation of the form

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

where k is a constant, prove that the conditions of the problem are satisfied by

$$T = \text{Real part of } F(x, y) \exp \left\{ i\gamma t - \sqrt{\frac{\pi}{2k}} (1 + i)z \right\}$$

44. A transmission line consists of two parallel extremely long wires having a resistance R ohms per mile, that is per mile of lead and return, a capacity C farads per mile, an induction L henrys per mile, and a dielectric conductance of S mhos per mile. Prove that the equations for the propagation of a current in the line are

$$\frac{\partial v}{\partial x} = Ri + L \frac{\partial i}{\partial t} \quad \frac{\partial i}{\partial x} = Sv + C \frac{\partial v}{\partial t}$$

where v is the potential difference between the wires at a distance x from the receiving end and i is the current at that point.

Solve the above equations by assuming that $i = I(x) \sin pt$, $v = V(x) \sin (pt + \theta)$, and show that when $LS = CR$ the general solution can be expressed in a form analogous to the general solution of d'Alembert's equation. (O. Heaviside.)

45. Solve the equation

$$x \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 0$$

by means of a power series in x , using the supplementary conditions

$$y = 1 \quad \frac{d^2 y}{dx^2} = 0 \quad \text{for } x = 0$$

46. Prove that the differential equation

$$\frac{d^2 u}{dx^2} + \left(\frac{1}{x} - 1 \right) \frac{du}{dx} + \left(\frac{a}{x} - \frac{n^2}{x^2} \right) u = 0$$

is satisfied by the following series

$$u_1 = \frac{x^n}{\Gamma(2n+1)} \left[1 + \frac{n-a}{2n+1} \frac{x}{1!} + \frac{(n-a)(n-a+1)}{(2n+1)(2n+2)} \frac{x^2}{2!} + \dots \right]$$

$$u_2 = \frac{x^{-n}}{\Gamma(1-2n)} \left[1 + \frac{n+a}{2n-1} \frac{x}{1!} + \frac{(n+a)(n+a-1)}{(2n-1)(2n-2)} \frac{x^2}{2!} + \dots \right]$$

Express the solution in terms of Bessel functions when $a = -\frac{1}{2}, 0, \frac{1}{2}, 1$ and -1 respectively. (H. G. Savidge.)

47. Prove that the roots of the equation

$$u^3 - 3u + 6z = 0$$

satisfy the differential equation

$$(9z^3 - 1) \frac{d^2 u}{dz^2} + 9z \frac{du}{dz} - u = 0$$

Find a power series which represents the solution which is zero when $z = 0$.

48. The small transverse vibrations of a string of beads at equal distances apart are governed by an equation of type

$$\frac{d^2 y_s}{dt^2} = k^2 (y_{s+1} + y_{s-1} - 2y_s)$$

where y_s is the displacement of the s th bead from its equilibrium position. Determine the possible periods of vibration on the assumption that the ends of the string are fixed, i.e. $y_0 = 0$, $y_n = 0$, for all values of t .

49. The string of beads in the last example is supposed to extend to infinity in both directions and the bead for which $s = 0$ is initially displaced through a distance ε , while all the other beads are in their zero positions, i.e.

$$\begin{array}{ccccccc} y_0 = \varepsilon & & y_1 = 0 & & y_2 = 0 & & \dots \\ & & y_{-1} = 0 & & y_{-2} = 0 & & \dots \end{array}$$

The system then begins to move; use the equation of Ex. 48 to prove that

$$\left[\frac{d^{2r} y_s}{dt^{2r}} \right]_{t=0} = 0 \quad \text{if } r < s$$

and find a power series for y_s . Hence show that

$$y_s = J_{2s}(2kt)$$

(W. R. Hamilton and T. H. Havelock.)

50. If in Ex. 17 the rate at which the eggs go bad is a linear function of the time and proportional to the number of eggs present, prove that the solution of the problem depends on a linear equation of Laplace's type, and solve this equation by means of a definite integral.

INDEX

- Abdank-Abakanowicz 288, 290
 Abel 276
 Abel's equation 288
 Adjoint differential expressions 53,
 280
 Ajello 287
 d'Alembert's equation 176
 Appell 212, 215, 275
 Approximate solution, method for
 227, 248
 Approximations, method of succes-
 sive 245

 Baker 245
 Ballif 100, 102
 Barnes 235, 267
 Bateman 159
 Bendixon 252
 Bernoulli's equation 21
 Bertrand 164
 Bessel's equation 76, 113, 198, 200,
 221, 237, 272
 Bilinear concomitant 53
 Birkhoff 255
 Bliss 245, 252
 Block 175
 Bôcher 253
 Bois Reymond 170
 Bolza 75, 79
 Boole 51, 62
 Branch point, movable 69
 Brill 212, 215
 Burkhardt 15

 Cailler 100
 Canonical form of a harmonic equa-
 tion 218
 Canonical form of a linear partial
 differential equation of the second
 order 170
 Canonical form of a total differential
 expression 164
 Carslaw 215, 256, 270
 Cauchy 91, 93, 245, 271, 272

 Cauchy's existence theorem 142, 223
 Cayley 92
 Characteristics 131, 172
 differential equations of 133, 171
 Chrystal 92, 94, 131
 Clairaut 50
 Clairaut's equation 63
 Cohen 83, 93
 Complementary function 54
 Complete integral 122
 Complete solution 122
 Constants, number of, in general
 solution 3
 Contact transformation 81, 148
 Continuity, equation of 139
 Convergence theorems 242
 Cusp-locus 90, 95

 Darboux 56, 90, 100, 101, 128, 174,
 216, 219, 221, 280, 286
 De Donder 159
 Degree of an equation 2
 Dependent variable 1
 Dirichlet's conditions 257, 270
 Dixon 159, 255
 Donkin 210
 Duality, principle of 81, 148
 Dyck, 93, 295

 Elliptic equation 170
 Emden 229
 Envelope 89, 92, 95
 Euler 50, 179, 261
 Euler's equation 103
 Euler's transformation 265
 Evans 285

 Forsyth 37, 129, 142, 164, 175, 184,
 204, 230, 235, 244
 Fouret 88
 Fourier 211
 Fourier's inversion formula 270
 Fréchet 279
 Frobenius 243

- Fuchs 243
 Fundamental solution 269

 Gaskin 115
 Gegenbauer 115
 General integral 122
 General solution 3
 Gevrey 285
 Goursat 49, 117, 142, 143, 281, 285
 Grade 80
 Green's function 15, 276, 283
 Guldberg 163

 Haar 255
 Hackett 8, 230
 Hargreaves 159, 192
 Harmonic equation 216, 270
 Harmonic function 193
 Harrison 229
 Hayashi 295
 Heaviside 284, 285
 Heine 265
 Hertz 186
 Heun 228, 229
 Heywood 279
 Hilbert 255, 276, 280
 Hill 92, 94, 96, 128
 Hobson 255, 265, 274
 Holmgren 285
 Homogeneous equation 80
 Hudson 92
 Hypergeometric equation 111, 170, 232
 Hypergeometric function 112, 241

 Indicial equation 231
 Integrability, condition of 153
 Integral curve 88
 Invariant 72, 76, 161
 Inversion 199, 201

 Jacobi 259
 Jacobian 116
 Jacobi's equation 83
 Jordan 265

 Kelvin 197, 199, 230, 287, 294
 Kirchhoff 284
 Kneser 255, 278
 Kummer 112, 236, 237
 Kutta 228, 229

 Lagrange's auxiliary equation 135
 linear equation 134
 type of equation 67
 Laguerre 110
 Lamb 197

 Lamé 197
 Laplace 170
 Laplace's equation 193, 271,
 method 173
 transformation 260
 type of equation 75
 Lebedeff 285
 Lecornu 125
 Legendre's equation 113, 198, 219,
 240, 259 266, 268
 Leibnitz 49, 91, 261, 273
 Levi 285
 Levi Civita 159, 204, 210
 Levy 128
 Lie 83
 Liénard 186
 Linear equation 17
 Lines of force 190
 Liouville 75
 Lipschitz's condition 249
 Lorentz 186
 Lovett 163

 Macdonald 268
 Mason 255
 Massau 94, 125, 248
 Maxwell's equations 179
 Melde 40
 Mellor 11
 Mercer 255
 Monge 124
 Morley 8
 Multipliers 136, 138

 Neumann 211
 Node-locus 93, 95

 d'Ocagne 248
 Order 2, 17
 Orthogonal trajectories 98

 Painlevé 249
 Parabolic equation 170, 285
 Particular integral 54
 Pascal 287, 288, 293
 Perron 98, 252
 Petrovitch 93, 96, 294
 Petzval 261
 Pfaff's problem 164
 type of equation 263
 Picard 70, 245, 284, 295
 Pochhammer 75, 237, 265
 Poincaré 284
 Poisson 46, 210, 274
 Potential function 193
 Primitive 3
 Pupin, 284

Rayleigh 38, 40, 230, 268
 Riccati's equation 68, 101
 Ricci 159
 Richardson 230
 Riemann 211, 280
 Righi 186
 Ritz 230
 Rouquet 102
 Runge 227
 Rutherford 42

Saint Robert 125
 Schafheitlin 73
 Scherk-Lobatto equation 77
 Schlömilch 75
 Scribanti 295
 Self-adjoint equation 54
 Separation of variables 10, 146
 Serret 86
 Silberstein 185
 Singular integral 122, 129
 solution 88, 90
 Special solution 140
 Stäckel 93

Standard forms 144
 Stefan 100
 Steimley 142
 Stephenson 38
 Stieltjes 201
 Stokes 115
 Symbolical differentiation 156
 method 33
 product 155

Tac-locus 89, 95, 97
 Taylor 91, 228
 Thomson, James, 288
 Thomson, J. J. 179, 190
 Turrière 140

Wave-equation 179
 Weber 185
 Wedderburn 94, 275
 Weight 80
 Weyr 70
 Whittaker 271
 Wilton 175
 Wright 159



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